Qualifying Exam 2018 Statistical Mechanics

Problem 1

The probability distribution of a particle's velocity in an ideal gas is given by

$$f(\vec{v}) = A \exp\left(-\frac{m\vec{v}^2}{2k_B T}\right)$$

a) Compute the normalization constant A in $D = 3$ dimensions.

b) Compute the most probable speed of a particle of the ideal gas

c) Compute the probability distribution $P(E)$ of a particle's energy in an ideal gas.

d) Compute the fraction of particles that exceed a certain speed v_g.

Problem 2

A model of a rubber band consists of a one-dimensional chain containing a large number N of linked rigid segments as shown in figure 1. Each segment is independent of the others. It occupies one of two possible states: horizontal, which contributes length a to the chain, or vertical, which contributes nothing to the length. The segments are linked so that they cannot come apart. The chain is in contact with a heat bath at temperature T.

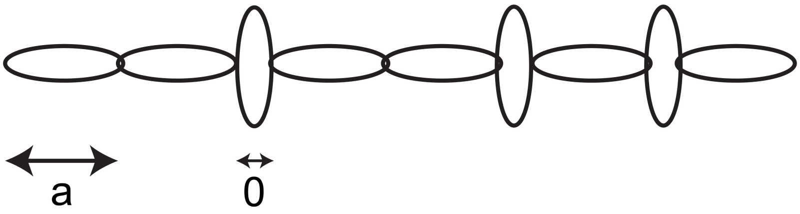

FIG. 1: One-dimensional chain containing a large number N of linked rigid segments.

a) If there is **no** energy difference between the the two states, what (expressed in terms of Na, and T) is the average length of the chain?

b) The chain is now fixed at one end to a wall and stretched horizontally at the other end by a weight with mass m hung over a pulley that supplies a horizontal force F. (See figure 2.) Determine the average length of the chain (again in terms of Na, and T) at any temperature. Find the average length in the limits that $T \to 0$ and $T \to \infty$.

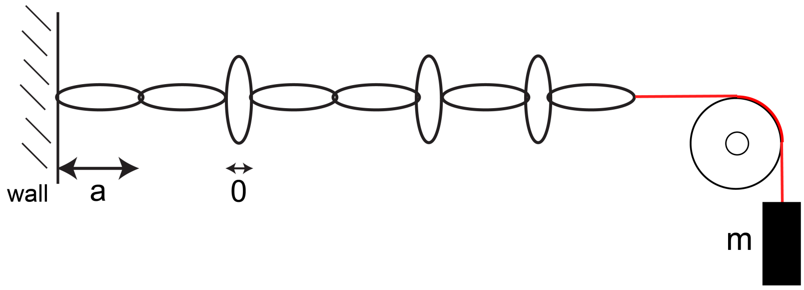

FIG. 2: One-dimensional chain fixed to a wall and and stretched horizontally at the other end by a weight with mass m hung over a pulley.

c) What is the requirement on temperature T and force F such that Hooke's Law (that the change in length from equilibrium is proportional to F) applies?

d) As the temperature is raised do we need more or less force to stretch the rubber band? If you warm up the band while applying a fixed force, will it expand or contract?

Problem 3

A spacecraft is in orbit around the Sun at distance R. It is shielded from the Sun's heat by a flat panel that is oriented perpendicular to the Sun and absorbs all of the incoming solar radiation. The Sun can be regarded a black body with surface temperature T_S, the radius of the sun is R_S, and the Boltzmann constant is given by σ.

a) Derive the formula giving the solar energy flux (the power per unit area) arriving at the panel.

b) The shielding panel is thermally insulated so that it only loses heat via photons re-radiated from its front surface. Assuming that the panel
can be treated as a black body, calculate the equilibrium temperature of the panel in terms of T_S, R_S and R.

c) The free energy of a volume V of black-body radiation is

$$F(T,V) = -\gamma V T^4$$

where γ is some constant that depends on k_B, $\hbar$ and the speed of light c (you do not need to compute it). Use a thermodynamic relationto express the internal energy U, the entropy S, and the pressure P of the volume of gas in terms of γ, V, and T. Hence find the dimensionless constant ζ such that $P = \zeta U/V$.

d) Assuming that the incoming energy flux (the answer to part (a)) is Q_{in}, compute P_{in}, the force per unit area due to the impact of the

photons on the panel. Compare your answer to the energy density in the incoming radiation. Does the constant from part (c) still apply?

If not why not?

Problem 4

Consider a model for n identical xenon atoms of mass m that are trapped on the surface of a solid. A xenon atom can be tightly bound to one of N adsorption sites with binding energy E_a (a positive number), or it may be free to move over the two-dimensional surface whose total area is A. When it is free to move along the surface, the xenon atom has both kinetic energy $\varepsilon_{kin} = mv^2/2 = p^2/2m$ and constant potential energy $\varepsilon_{pot} = V$. You may assume that the number n of xenon atoms is much smaller than the number of adsorption sites N, and that each adsorption site can bind at most one atom.

a) Consider first the two-dimensional gas composed of the n_g atoms that are free to move. Evaluate the partition function of the gas as a function of $n_g, A, m, \hbar, k_B$ and the temperature T. Also compute the free energy $F_g = E - TS$ and the chemical potential

$$\mu_g = \frac{\partial F_g}{\partial n_g}$$

as a function of the gas density n_g/A. (You may assume that the temperature is high enough that you may use Boltzmann statistics.)

b) Now consider the $n_b = n - n_g$ atoms that are bound to the adsorption sites. Find the mean energy, entropy, free-energy and chemical potential for these particles.

c) Show that in equilibrium the number n_g is determined by $n_b + n_g = n$ together with

$$\frac{n_b A}{n_g N} = C(T) \exp\left(\frac{E_a + V}{k_B T}\right)$$

You should express the function $C(T)$ in terms of $T, m, \hbar$ and k_B.

Problem 5

Consider a paramagnetic material consisting of a volume V of N non-interacting spin-1 particles with magnetic dipole moment

$$\vec{\mu} = \frac{\mu}{\hbar}\vec{S} \qquad S_z = m\hbar \qquad m = 0, \pm 1$$

that are located in a magnetic field $\vec{B} = B_0\hat{z}$ at temperature T.

(a) Write the partition function in terms of k_BT and $\varepsilon = \mu B_{0.}$

b) Compute the average spin $\langle S_z \rangle$.

(c) Calculate the average energy $\langle E \rangle$ of the N spins, and find the approximate functional form of $\langle E \rangle$ in the low- and high-temperature limits. Provide a qualitative argument to explain your results in these two limits. What is the relation between $\langle E \rangle / N$ and $\langle S_z \rangle$? Explain