① $V(r)=\begin{cases} V_0, & r<r_0 \\ 0, & r>r_0 \end{cases}$, $V_0>0$

a) $$f=-\frac{m}{2\pi\hbar^2}\int d^3\vec{r}'\,V(r')\,e^{i\vec{q}\cdot\vec{r}'}$$

$$=-\frac{m}{2\pi\hbar^2}\int_0^{2\pi}d\phi\int_{-1}^{1}d\cos\theta\int_0^{r_0}dr'\,r'^2\,V_0\,e^{iqr'\cos\theta}$$

$$=-\frac{mV_0}{\hbar^2}\int_0^{r_0}dr'\,r'^2\,\frac{2\sin qr'}{qr'}$$

$$=-\frac{2mV_0}{\hbar^2 q}\int_0^{r_0}dr'\,r'\sin qr'$$

$$=-\frac{2mV_0}{\hbar^2 q^3}\left(\sin qr_0-qr_0\cos qr_0\right)$$

$$\frac{d\sigma}{d\Omega}=|f|^2=\frac{4m^2V_0^2}{\hbar^4 q^6}\left(\sin qr_0-qr_0\cos qr_0\right)^2$$

b) $|q|\to 0$, so $e^{i\vec{q}\cdot\vec{r}'}\simeq 1$. Take $q\to 0$ in a) above.

$$f=-\frac{m}{2\pi\hbar^2}\int_0^{r_0}4\pi r'^2\,dr'\,V_0=-\frac{2mV_0r_0^3}{3\hbar^2}$$

and $$\frac{d\sigma}{d\Omega}=\frac{4m^2V_0^2r_0^6}{9\hbar^4}$$

c) $\frac{d\sigma}{d\Omega} = 10^{-12}\ cm^2$.

$$\frac{dN}{dt} = (\text{flux}) \cdot \frac{d\sigma}{d\Omega} \cdot d\Omega$$

$$\simeq \left(10^{16}\ \frac{\text{particles}}{cm^2}\right)\left(10^{-12}\ cm^2\right)\left(10^{-4}\right) = 1 \text{ particle/s}$$

where $d\Omega \simeq 10^{-4}$ since detector area of $1\,cm^2$ was about 10^{-5} of the total surface area at a distance of 100cm. Scattering angle is not relevant since the particles are slow.

(2) Write Schrödinger eqn for ψ_{n_1} & ψ_{n_2}:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{n_1}}{dx^2} + U(x)\psi_{n_1} = E_{n_1}\psi_{n_1} \quad (1)$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{n_2}}{dx^2} + U(x)\psi_{n_2} = E_{n_2}\psi_{n_2} \quad (2)$$

$$(1)\times\psi_{n_2} - (2)\times\psi_{n_1} \Rightarrow \quad -\frac{\hbar^2}{2m}\psi_{n_2}\frac{d^2\psi_{n_1}}{dx^2} + \frac{\hbar^2}{2m}\psi_{n_1}\frac{d^2\psi_{n_2}}{dx^2} = \left(E_{n_1} - E_{n_2}\right)\psi_{n_1}\psi_{n_2}$$

or

$$\frac{\hbar^2}{2m}\frac{d\left(\frac{d\psi_{n_1}}{dx}\psi_{n_2} - \frac{d\psi_{n_2}}{dx}\psi_{n_1}\right)}{dx} = \left(E_{n_2} - E_{n_1}\right)\psi_{n_1}\psi_{n_2} \quad (3)$$

Suppose ψ_{n_1} has two consecutive nodes at x_1 and x_2. WLOG assume ψ_{n_1} is positive between the two zeros. Now do proof by contradiction: assume ψ_{n_2} has no nodes between x_1 and x_2. WLOG assume ψ_{n_2} is positive between x_1 and x_2.

Integrate (3) between x_1 ~~and~~ to x_2

$$\frac{\hbar^2}{2m}\left[\frac{d\psi_{n_1}}{dx}\psi_{n_2} - \frac{d\psi_{n_2}}{dx}\psi_{n_1}\right]\Bigg|_{x_1}^{x_2} = \left(E_{n_2}-E_{n_1}\right)\int_{x_1}^{x_2} dx\,\psi_{n_1}\psi_{n_2}.$$

$$\frac{\hbar^2}{2m}\Bigg[\frac{d\psi_{n_1}(x_2)}{dx}\psi_{n_2}(x_2) - \frac{d\psi_{n_2}(x_2)}{dx}\cancelto{0}{\psi_{n_1}(x_2)}$$

$$-\frac{d\psi_{n_1}(x_1)}{dx}\psi_{n_2}(x_1) + \frac{d\psi_{n_2}(x_1)}{dx}\cancelto{0}{\psi_{n_1}(x_1)}\Bigg] = \left(E_{n_2}-E_{n_1}\right)\int_{x_1}^{x_2} dx\,\psi_{n_1}\psi_{n_2}$$

or

$$\frac{\hbar^2}{2m}\left[\frac{d\psi_{n_1}(x_2)}{dx}\psi_{n_2}(x_2) - \frac{d\psi_{n_1}(x_1)}{dx}\psi_{n_2}(x_1)\right] = \left(E_{n_2}-E_{n_1}\right)\int_{x_1}^{x_2} dx\,\psi_{n_1}\psi_{n_2}.$$

Now $\frac{d\psi_{n_1}(x_1)}{dx} > 0$ and $\frac{d\psi_{n_1}(x_2)}{dx} < 0$, so LHS is negative definite but RHS is positive definite. $\Rightarrow\Leftarrow$

So $\psi_{n_2}(x)$ must have at least one node between x_1 and x_2.

3. Two fermions w/ spin up means spin wavefunction χ is symmetric, so spatial wavefunction must be antisymmetric.

a) $A=0$:

$$\Psi = \psi(\phi_1, \phi_2)\chi_{\uparrow\uparrow} = \frac{1}{\sqrt{2}}\left(\psi_i(\phi_1)\psi_j(\phi_2) - \psi_i(\phi_2)\psi_j(\phi_1)\right)$$
$$\times \chi_{\uparrow\uparrow}$$

where i and j specify single particle eigenstates

Single particle $H = \frac{L_z^2}{2ma^2} = -\frac{\hbar^2}{2ma^2}\frac{\partial^2}{\partial\phi^2}$

$H\psi = E\psi$ eigenstates are $\psi_n(\phi) = \frac{1}{\sqrt{2\pi}} e^{in\phi}$

$$\int_0^{2\pi} \psi_n^* \psi_n \, d\phi = 1$$

Two particle $H = \frac{L_{z1}^2}{2ma^2} + \frac{L_{z2}^2}{2ma^2}$

$$= -\frac{\hbar^2}{2ma^2}\left(\frac{\partial^2}{\partial\phi_1^2} + \frac{\partial^2}{\partial\phi_2^2}\right).$$

$H\Psi = E\Psi$ eigenstates in general are

$$\Psi_{nn'} = \frac{1}{\sqrt{2}}\frac{1}{2\pi}\left[e^{in\phi_1}e^{in'\phi_2} - e^{in'\phi_1}e^{in\phi_2}\right]\chi_{\uparrow\uparrow}$$

which requires $n \neq n'$

$$H\Psi_{nn'} = \underbrace{\frac{\hbar^2}{2ma^2}(n^2+n'^2)}_{= E_{nn'}}\Psi_{nn'}$$

(b) Ground state: $n'^2+n^2 = 1$, $E = \hbar^2/2ma^2$

Possible values of (n,n') are $(0,1)$, $(1,0)$, $(0,-1)$ $(-1,0)$.

There are two-fold degenerate, since $(0,1)$ and $(1,0)$, (and $(0,-1)$ and $(-1,0)$) are same to within overall phase factor

1st excited state: $(n')^2+n^2 = 2$, $E = \frac{\hbar^2}{ma^2}$

Possible (n,n') are $(1,-1)$, $(-1,1)$

Two-fold degenerate.

Now $A \neq 0$, but small

(d) $\Delta E = \langle nn' | \hat{V} | nn' \rangle$ (off diagonal are zero — see next page)

Consider ground state $|10\rangle_{nn'}$

$$= \frac{1}{\sqrt{2}\,2\pi}\left(e^{i\phi_1} - e^{i\phi_2}\right)$$

$$\langle nn' | \hat{V} | nn' \rangle = \frac{A}{8\pi^2} \int d\phi_1 \, d\phi_2 \cos(\phi_1 - \phi_2) \times \Big(\underbrace{2}_{\text{integral goes to zero}} \underbrace{- e^{i(\phi_1-\phi_2)} - e^{i(\phi_2-\phi_1)}}_{-2\cos(\phi_1-\phi_2)} \Big)$$

$$= -\frac{A}{8\pi^2} \underbrace{\int_0^{2\pi} d\phi_1 \, d\phi_2 \, 2\cos^2(\phi_1 - \phi_2)}_{= \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi \, 2\cos^2\phi \; = 2\pi \cdot 2 = 4\pi}$$

$$= -\frac{A}{2}$$

Same for all other ground states.

~~1st excited state, $|1-1\rangle$~~

$$\cancel{\langle nn' | \hat{V} | nn' \rangle = \frac{A}{8\pi^2} \int d\phi_1 \, d\phi_2 \cos(\phi_1 - \phi_2) \times}$$

$\cancel{(}$

Consider off-diagonal matrix element

Sample:

$\langle 0\ -1 | \hat{V} | 1\ 0 \rangle$

$$= \frac{A}{8\pi^2} \int d\phi_1 d\phi_2 \cos(\phi_1 - \phi_2)\left(e^{i(\phi_1-\phi_2)} e^{-i(\phi_1-\phi_2)} - 2 \right)$$

$$= \frac{A}{8\pi^2} \int d\phi_1 d\phi_2 \cos(\phi_1 - \phi_2)\left(2\sin(\phi_1 - \phi_2) - 2 \right)$$

$= 0$ (odd integrand)

eg $\sin(\phi_1 - \phi_2)\cos(\phi_1 - \phi_2) = \frac{1}{2}\sin 2(\phi_1 - \phi_2)$

4) Unperturbed: $\langle\uparrow|H_0|\uparrow\rangle = -\gamma B \langle \begin{smallmatrix}1\\0\end{smallmatrix}| \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}|\begin{smallmatrix}1\\0\end{smallmatrix}\rangle = -\frac{\gamma B\hbar}{2}$

$\langle\downarrow|H_0|\downarrow\rangle = -\gamma B \langle \begin{smallmatrix}0\\1\end{smallmatrix}| \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}|\begin{smallmatrix}0\\1\end{smallmatrix}\rangle = +\frac{\gamma B\hbar}{2}$

So $E_f - E_i = \gamma B\hbar$

$$C_f^{(1)}(t) = \frac{1}{i\hbar}\int_{-\infty}^{t} \langle f|H'|i\rangle e^{i(E_f-E_i)t/\hbar}\,dt$$

$$C_f^{(1)}(\infty) = \frac{1}{i\hbar}\int_{-\infty}^{\infty} (-\gamma b)e^{-i\omega_p t}\langle\downarrow|\hat{S}_x|\uparrow\rangle e^{-t^2/\tau^2} e^{i(E_f-E_i)t/\hbar}\,dt$$

$$= \frac{-\gamma b}{i\hbar}\langle\downarrow|\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}|\uparrow\rangle \int_{-\infty}^{\infty} e^{i(\omega_{fi}-\omega_p)t} e^{-t^2/\tau^2}\,dt$$

where $\omega_{fi} = \frac{E_f - E_i}{\hbar} = B\gamma$

Consider $\int_{-\infty}^{\infty} e^{i(\omega_{fi}-\omega_p)t} e^{-t^2/\tau^2}\,dt$

$$= \int_{-\infty}^{\infty} e^{-(t-\frac{i}{2}(\omega_{fi}-\omega_p)\tau^2)^2/\tau^2}\,dt \cdot e^{-(\omega_{fi}-\omega_p)^2\tau^2/4}$$

$$= \sqrt{\pi}\,\tau\, e^{-(\gamma B-\omega_p)^2\tau^2/4}$$

So, $C_f^{(1)}(\infty) = \frac{-\gamma b}{i\hbar} \cdot \frac{\hbar}{2} \cdot \sqrt{\pi}\,\tau \; e^{-(\gamma B - \omega_p)^2 \tau^2/4}$

$$\text{Probability} = \left| C_f^{(1)}(\infty) \right|^2$$

$$= \frac{\pi \gamma^2 b^2 \tau^2}{4} \, e^{-(\gamma B - \omega_p)^2 \tau^2/2}$$

- Largest when $\omega_p = \gamma B$ (classical precession frequency)

(5) For any function of electron energy, $F(\mathcal{E}(\vec{k}))$, total value is

$$\underbrace{2}_{\text{spin}} \sum_k F(\mathcal{E}(k)) = 2\frac{L}{2\pi}\int_{-\infty}^{\infty} dk\, F(\mathcal{E}(k))$$

$$\mathcal{E} = \frac{\hbar^2 k^2}{2m}, \quad d\mathcal{E} = \frac{\cancel{2}\hbar^2 k\, dk}{\cancel{2}m}$$

$$= \frac{L}{\pi}\underbrace{2}_{\text{from } \pm k}\int_0^{\infty} \frac{d\mathcal{E}\, m}{\hbar^2\sqrt{\frac{2m\mathcal{E}}{\hbar^2}}} F(\mathcal{E}(k))$$

$$= \frac{2Lm}{\pi\hbar^2}\sqrt{\frac{\hbar^2}{2m}}\int \frac{d\mathcal{E}}{\sqrt{\mathcal{E}}} F(\mathcal{E}(k))$$

$\psi(x) \sim e^{ikx}$

$\psi(x+L) = \psi(x)$

$\Rightarrow e^{ik(x+L)} = e^{ikx}$

$e^{ikL} = 1$

$L = \frac{\overset{\text{integer}}{n}\cdot 2\pi}{k}$

or

$k = \frac{2\pi n}{L}$

So

$$D(\mathcal{E}) = \frac{2m}{\pi\hbar^2}\sqrt{\frac{\hbar^2}{2m}}\frac{1}{\sqrt{\mathcal{E}}}$$

$$= \frac{2}{\pi}\sqrt{\frac{m}{2\hbar^2\mathcal{E}}}$$

- singular at $\mathcal{E} = 0$

$$N = 2\left(\frac{L}{2\pi}\right)2k_F \Rightarrow k_F = \frac{N\pi}{2L} \Rightarrow \mathcal{E}_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2 N^2\pi^2}{8L^2 m}$$