University of Illinois at Chicago
Department of Physics

Quantum Mechanics
Preliminary Examination

January 8, 2018
9:00 am – 12:00 pm

Possibly useful formulas:

$\int x \sin ax \; dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax$

$\int_0^\infty e^{-a^2x^2} \, dx = \frac{\sqrt{\pi}}{2a}$

$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$c_f^{(1)}(t) = \frac{1}{i\hbar} \int_{-\infty}^{t} \langle f | \hat{H}' | i \rangle e^{i(E_f - E_i)t/\hbar}$

$$\int_0^\infty x^{2n} e^{-ax^2} \, dx = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}, \;\; n > 0$$

$$f \cong -\frac{m}{2\pi\hbar^2} \int e^{i(\boldsymbol{k}' - \boldsymbol{k}) \cdot \boldsymbol{r}} V(\boldsymbol{r}) d^3\boldsymbol{r}$$

Full credit can be achieved from completely correct answers to <u>4 questions</u>. If the student attempts all 5 questions, all of the answers will be graded, and the <u>top 4 scores</u> will be counted toward the exam's total score.

1. Particles are scattered by a potential $V(r) = V_o$ if $r < r_o$ and $V(r) = 0$ if $r > r_o$, with $V_o > 0$.
   (a) Find the differential cross section in the Born approximation.
   (b) What is the limit of your answer to (a) when the particles are very slow?
   (c) Suppose $\frac{d\sigma}{d\Omega}$=10^{-12} cm^2. If 10^{16} particles per cm^2 per second are incident on this target at low speeds, approximately (i.e. to nearest order of magnitude) how many per second are detected on a 1 cm^2 detector at 90^o from the incident direction 1 m ($\gg r_o$) from the target?

2. Consider bound states $\psi_n(x)$ of energy E_n in an *arbitrary* (i.e. not necessarily square well) one-dimensional potential well. Any state will have at least two nodes (for example at $x \to \pm\infty$). Prove that between two consecutives nodes of ψ_{n_1}, there is at least one node of ψ_{n_2} where $E_{n_2} > E_{n_1}$. *Hints:* Combine the Schrödinger equations for ψ_{n_1} and ψ_{n_2} in such a way as to eliminate the potential terms, and use proof by contradiction.

3. Consider a simplified model to calculate the electronic structure of a ring-shaped molecule. Take two identical fermions of mass *m* and spin 1/2, moving on a ring of radius a, with spins constrained to point *up*. The particles can interact through a potential of form

$$V(\phi_1, \phi_2) = A cos(\phi_1 - \phi_2)$$

   where ϕ_1 and ϕ_2 are the angular positions of the two particles.

   First assume $A = 0$.

   (a) The only dynamics in this system are the constrained motion (kinetic energy) of a particle going in a circle. The Hamiltonian may therefore be written as a rigid rotator $H = \frac{L_z^2}{2mr^2}$. Find the allowed energy levels of this system of two electrons.
   (b) Find the ground and 1st excited state wave functions.
   (c) Determine the degeneracies of the ground and 1st excited states.

   Now assume A is nonzero but weak, $A \ll \hbar^2/(2ma^2)$.

   (d) Find the energies and degeneracies of the ground state in part (c) to first order in A.

4. A spin-1/2 particle (e.g., an electron) is in a constant magnetic field B along the z-axis, yielding an unperturbed initial Hamiltonian $H_o = -\gamma B S_z$. Initially (at $t = -\infty$), it is in the eigenstate $|\uparrow\rangle$ of this Hamiltonian, with $S_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle$. Now we slowly turn on and then back off a weak oscillating magnetic field in the x-direction, yielding a perturbing Hamiltonian $H' = -\gamma b e^{-i\omega_p t} S_x e^{-t^2/\tau^2}$. Calculate the probability that after some very long time ($t = +\infty$), the electron will be in the state $|\downarrow\rangle$, as a function of all parameters given. For very large time constants τ, what is the condition between ω_p and B that yields the largest transition probability?

5. Consider a zero temperature Fermi gas of N electrons (each of mass m) in a one-dimensional system of length L. Assume the wave function satisfies periodic boundary conditions at the edges of the system. Express the density of states $D(\varepsilon)$ per unit energy ε (including both spin orientations), and the Fermi energy ε_F (the energy of the highest occupied state), in terms of N, L and m.