(1) $-\vec{\nabla}^2\phi = \rho/\varepsilon_0$

since $-(\cos\alpha x)'' = \alpha^2\cos\alpha x$, try solution in the form:

$$\phi = \phi_0 \cos\alpha x \cos\beta y \cos\gamma z$$

$$-\vec{\nabla}^2\phi = \phi_0(\alpha^2+\beta^2+\gamma^2)\cos\alpha x \cos\beta y \cos\gamma z = \frac{\rho_0}{\varepsilon_0}\cos\alpha x \cos\beta y \cos\gamma z$$

$$\Rightarrow \phi_0 = \frac{\rho_0/\varepsilon_0}{\alpha^2+\beta^2+\gamma^2} \Rightarrow \boxed{\phi = \frac{\rho_0/\varepsilon_0}{\alpha^2+\beta^2+\gamma^2}\cos\alpha x \cos\beta y \cos\gamma z}$$

(2) $U = q\,\phi(0)$

To find $\phi(r)$ first find $\vec{E}$ using Gauss' law:

$$E_r \cdot 4\pi r^2 = \frac{\rho_0}{\varepsilon_0}\frac{4\pi r^3}{3} = \frac{Q}{\varepsilon_0}\frac{r^3}{R^3}, \quad r<R \qquad \left(\rho_0 = \frac{3Q}{4\pi R^3}\right)$$

$$E_r = \frac{Q}{4\pi\varepsilon_0}\begin{cases}\frac{r}{R^3}, & r<R \\ \frac{1}{r^2}, & r>R\end{cases}$$

$$\phi(0) = \int_0^\infty E_r\,dr = \frac{Q}{4\pi\varepsilon_0}\left\{\int_0^R \frac{r}{R^3}dr + \int_R^\infty \frac{1}{r^2}dr\right\} = \frac{Q}{4\pi\varepsilon_0 R}\left(\frac{1}{2}+1\right)$$

$$= \frac{3Q}{8\pi\varepsilon_0 R} \Rightarrow \boxed{U = \frac{3qQ}{8\pi\varepsilon_0 R}}$$

Alternatively:

$$U = \int d^3r\,\rho(\vec{r})\,\phi_q(\vec{r}) = \int d^3r\,\rho\,\frac{q}{4\pi\varepsilon_0|\vec{r}|} = \frac{\rho_0 q}{\varepsilon_0}\int_0^R dr\,r^2\cdot\frac{1}{r} = \frac{\rho_0 q}{\varepsilon_0}\frac{R^2}{2}$$

$$= \frac{3qQ}{8\pi\varepsilon_0 R}$$

(3) $\vec{\nabla}\cdot\vec{D}=\rho$; $\vec{D}=\varepsilon\vec{E}$; $\varepsilon=\varepsilon_0\frac{x+d}{d}$

Inside dielectric: $\rho=0$.

$$\vec{\nabla}(\varepsilon\vec{E})=0 \Rightarrow \varepsilon\vec{E}=\text{const} \Rightarrow E_x=\frac{\overset{\text{a constant}}{\alpha}}{\varepsilon} \quad (E_y=E_z=0 \text{ by symmetry})$$

$$V=\Phi(0)-\Phi(d)=\int_0^d E_x dx=\frac{\alpha d}{\varepsilon_0}\int_0^d\frac{dx}{x+d}=\frac{\alpha d}{\varepsilon_0}\ln 2 \Rightarrow \alpha=\frac{\varepsilon_0 V}{d\ln 2}$$

$Q=\sigma A$; Gauss' law: $\sigma=D_x\Big|_{x=0}=\varepsilon E_x\Big|_{x=0}=\alpha=\frac{\varepsilon_0 V}{d\ln 2}$

$$C=\frac{Q}{V}=\frac{\sigma A}{V}=\boxed{\frac{\varepsilon_0 A}{d\ln 2}}$$

Bulk bound charge:

$$\rho_B=-\vec{\nabla}\cdot\vec{P}=-\vec{\nabla}(\vec{D}-\varepsilon_0\vec{E})=\varepsilon_0\vec{\nabla}\cdot\vec{E}=\varepsilon_0\frac{d}{dx}\frac{\alpha}{\varepsilon}=$$

$$=\alpha\frac{d}{dx}\left(\frac{d}{x+d}\right)=-\frac{\alpha d}{(x+d)^2}=\boxed{-\frac{\varepsilon_0 V}{\ln 2}\cdot\frac{1}{(x+d)^2}}$$

Boundary bound charges:

$$\sigma_B\Big|_{x=0}\equiv\sigma_B(0)=-[\vec{n}\cdot\vec{P}]_{\text{discontinuity}}=-P_x(x+0)+\underbrace{P_x(x-0)}_{=0}=-P_x(0)$$

$$\sigma_B\Big|_{x=d}\equiv\sigma_B(d)=-\underbrace{P_x(d+0)}_{=0}+P_x(d-0)=+P_x(d)$$

$$P_x=(\vec{D}-\varepsilon_0\vec{E})_x=(\varepsilon-\varepsilon_0)E_x=\alpha\left(1-\frac{\varepsilon_0}{\varepsilon}\right)=\alpha\left(1-\frac{d}{x+d}\right)=\alpha\frac{x}{x+d}$$

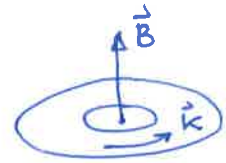
$$\sigma_B(0)=-P_x(0)=\boxed{0}$$

$$\sigma_B(d)=\frac{\alpha}{2}=\boxed{\frac{\varepsilon_0 V}{2d\ln 2}}$$

Check total bound charge (per area)

$$\sigma_B(0)+\sigma_B(d)+\int_0^d dx\,\rho_B=\frac{\alpha}{2}-\int_0^d\frac{\alpha d}{(x+d)^2}dx=0$$

④ $I = \int_a^b K_0\, dr = \boxed{K_0(b-a)}$



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{B}(0) = -\frac{\mu_0}{4\pi}\int d^3r \, \frac{\vec{J}\times\vec{r}}{|\vec{r}|^3} = -\frac{\mu_0}{4\pi}\int \underset{\text{area integral}}{da} \, \frac{\vec{K}\times\vec{r}}{|\vec{r}|^3}$$

$$\left\{ \hat{\varphi}\times\hat{r} = -\hat{z} \right.$$

$$= \frac{\mu_0}{4\pi} K_0 \hat{z} \int da \, \frac{1}{r^2} = \frac{\mu_0}{\cancel{4\pi}_{2}} K_0 \hat{z} \cdot \cancel{2\pi}\int_a^b dr\, r \frac{1}{r^2} = \boxed{\hat{z}\cdot \frac{\mu_0 K_0}{2} \cdot \ln\frac{b}{a}}$$

$$\vec{m} = \frac{1}{2}\int d^3r\, \vec{r}\times\vec{J} = \frac{1}{2}\int da\, \vec{r}\times\vec{K} = \frac{1}{2}K_0\hat{z}\int da\, r =$$

$$= \frac{1}{\cancel{2}} K_0 \hat{z}\cdot \cancel{2}\pi \int_a^b dr\, r^2 = \boxed{\hat{z}\cdot K_0 \frac{\pi(b^3-a^3)}{3}}$$

Check: $b \to a \quad m \to I\pi a^2 \quad B \to \frac{\mu_0 I}{2}$

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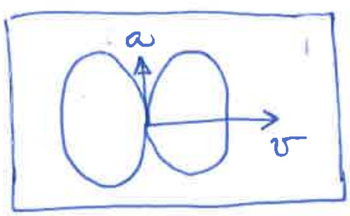
⑤ $a = F/m = q_e E/m_e \; ; \; t = \frac{\ell}{v} \; ; \; E = V/d$

$$at \ll v \Rightarrow a\ell \ll v^2 \Rightarrow v \gg \sqrt{a\ell} = \sqrt{\frac{q_e E \ell}{m_e}} \Rightarrow \boxed{v \gg \sqrt{\frac{q_e V \ell}{d m_e}}}$$

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \; ;$$

$$W = P\cdot t = \boxed{\frac{\mu_0 q_e^2}{6\pi c}\cdot\left(\frac{q_e V}{d m_e}\right)^2 \cdot \frac{\ell}{v}}$$



$\frac{dP}{d\Omega} = 0$ along the axis of acceleration: $\boxed{\text{up and down}}$