Solutions

1. (a) The two parallel springs is equivalent to a single spring with a spring constant $2k$. The total weight on this spring is the sum of the two masses, that is, $2Mg$. Therefore, the equilibrium positions are $x_1^{eq} = \frac{2Mg}{2k} = \frac{Mg}{k}$, and $x_2^{eq} = \frac{Mg}{k}$.

   (b) The kinetic energy is $T = \frac{1}{2}M\dot{X}_1^2 + \frac{1}{2}M(\dot{X}_1 + \dot{X}_2)^2$. The potential energy is a sum of the energy stored in the springs and the gravitational energy. The former is $V_1 = \frac{1}{2}(2k)x_1^2 + \frac{1}{2}kx_2^2 = \frac{1}{2}(2k)(x_1^{eq} + X_1)^2 + \frac{1}{2}k(x_2^{eq} + X_2)^2$, and the latter is $V_2 = -Mgx_1 - Mg(x_1 + x_2) = -Mg(x_1^{eq} + X_1) - Mg(x_1^{eq} + x_2^{eq} + X_1 + X_2)$. Using x_1^{eq} and x_2^{eq}, it is easy to check that linear terms in X_1 and X_2 cancel in the total potential energy $V = V_1 + V_2$, and we have $V = kX_1^2 + \frac{1}{2}kX_2^2 + \text{constant}$. The Lagrangian is therefore $L = T - V = \frac{1}{2}M\dot{X}_1^2 + \frac{1}{2}M(\dot{X}_1 + \dot{X}_2)^2 - kX_1^2 - \frac{1}{2}kX_2^2$ up to a constant.

   (c) From the Lagrange's equation of motion, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_i}\right) = \frac{\partial L}{\partial X_i}$, we get

$$M\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{pmatrix} = -k\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

   (d) The normal modes are obtained by replacing $\cdot$ with $-i\omega$, so the above equation becomes

$$\begin{pmatrix} -2M\omega^2 + 2k & -M\omega^2 \\ -M\omega^2 & -M\omega^2 + k \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   A solution exists only if the determinant of the above matrix is zero, and we have the two solutions as $\omega^2 = (2 \pm \sqrt{2})\frac{k}{M}$. The upper sign corresponds to the out-of phase oscillation of the two masses, and the lower sign corresponds to the in-phase oscillation. The normal modes are

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} -2 \mp \sqrt{2} \\ 2 \pm 2\sqrt{2} \end{pmatrix}.$$

2. (a) The impulse is equal to the momentum change of the center of mass, so the center of mass velocity after the impulse is $v_{CM} = \frac{P}{M}$. The impulse times the length from the center of mass ($\frac{l}{2}$) is equal to the change of angular momentum about the center of mass, $I_{CM}\omega = \frac{1}{12}Ml^2\omega$. Therefore, the angular velocity after the impulse is $\omega = \frac{6P}{Ml}$.

   (b) The velocity at the other end is given by a sum of the center of mass velocity $v_{CM} = \frac{P}{M}$ and the rotational velocity $-\frac{l}{2}\omega = -\frac{3P}{M}$, where the negative sign indicates that its direction is opposite to the original impulse. We get $v = -\frac{2P}{M}$, and the negative sign indicates that its direction is opposite to the original impulse.

   (c) Let P' be the magnitude of the exchanged impulse. For the rod, the total impulse is $P + P'$, so the CM velocity is $v_{CM} = \frac{P+P'}{M}$. The total change of angular momentum is however given by the difference, $I_{CM}\omega = \frac{l}{2}(P - P')$, and we have $\omega = \frac{6}{Ml}(P - P')$. Therefore, the velocity at the other end is $v = v_{CM} - \frac{l}{2}\omega = \frac{P+P'}{M} - \frac{3(P-P')}{M} = \frac{-2P+4P'}{M}$. On the other hand, the velocity of the bob of the mass is $-\frac{P'}{m}$ (the negative sign comes from the fact that the exchanged impulse on the bob of mass is opposite to that on the rod). Equating this with v, we find $P' = \frac{2m}{M+4m}P$ and $v = -\frac{2P}{M+4m}$.

   (d) From the above result, it is straight forward to compute $v_{CM} = \frac{P+P'}{M} = \left(\frac{M+6m}{M+4m}\right)\frac{P}{M}$, where $\frac{P}{M}$ is the result without the bob of mass. Therefore the CM velocity is larger than the case without the bob of mass. On the other hand, the angular velocity is $\omega = \frac{6}{Ml}(P - P') = \frac{6P}{Ml}\left(\frac{M+2m}{M+4m}\right)$ where $\frac{6P}{Ml}$ is the result without the bob of mass, so the angular velocity is smaller.

3. (a) The energy conservation equation reads as $(p_1 + p_2)c + m\gamma c^2 = Mc^2$, and the momentum conservation equation is $p_1 + m\gamma v = p_2$.

(b) Removing p_2, we get an equation

$$p_1 = \frac{c}{2}\left(M - m\gamma\left(1 + \frac{v}{c}\right)\right) = \frac{c}{2}\left(M - m\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\right) .$$

Imposing the condition $p_1 \geq 0$, we obtain

$$v \leq c\frac{\left(\frac{M}{m}\right)^2 - 1}{\left(\frac{M}{m}\right)^2 + 1} \equiv v_{max} .$$

(c) The energy conservation equation is the same $(p_1 + p_2)c + m\gamma c^2 = Mc^2$, and the momentum conservation equation becomes $p_1 + p_2 = m\gamma v$. Removing $(p_1 + p_2)$, we get an equation

$$m\gamma\left(1 + \frac{v}{c}\right) = m\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = M . \tag{0.1}$$

Solving this for v, we get $v = c\frac{\left(\frac{M}{m}\right)^2 - 1}{\left(\frac{M}{m}\right)^2 + 1} = v_{max}$.

(d) Since the net momentum should always be zero, the final nucleus can get the maximum recoil momentum when the momentum of the electron pair is maximal. Clearly this happens when the two electrons move to the same direction. Therefore the velocity in problems (b) and (c) should be the maximum velocity of the final nucleus.

4. (a) The reduced mass is $\frac{1}{\mu} = \frac{1}{M} + \frac{1}{M} = \frac{2}{M}$, or $\mu = \frac{M}{2}$. Therefore, the equation of motion for $\vec{R}$ with the Newton's gravitational force is $\mu\ddot{\vec{R}} = -\frac{GM^2}{R^2}\frac{\vec{R}}{R}$, which is equivalent to $\ddot{\vec{R}} = -\frac{2GM}{R^2}\frac{\vec{R}}{R}$. This is the equation of motion for a test particle moving in the gravitational force field produced by a mass $2M$.

(b) It is easy to derive the Kepler's formula $T = (2\pi)\sqrt{\frac{R^3}{2GM}}$.

(c) We can write the period as $T = (2\pi)\sqrt{\frac{R_\odot^3}{GM_\odot}} \times \sqrt{\frac{(R/R_\odot)^3}{2(M/M_\odot)}}$, where the first factor is the period of the Earth's orbit around the Sun which is 3.2×10^7 seconds. Putting $R/R_\odot = 10^{-6}$ and $2(M/M_\odot) = 2.8$, we have $T = 3.2 \times 10^7 \times \sqrt{\frac{10^{-18}}{2.8}} \approx 3.2/1.7 \times 10^{-2} \approx 2 \times 10^{-2}$ seconds. The frequency is then $1/T = 50$ Hz.

(d) Since the coordinates of the second star is precisely the minus of those of the first star, it is easy to see that the contribution from each star to Q_{ij} is the same, so the net quadrupole moment is twice of that of the first star. By definition, we have $Q_{xx} = 2 \times Mx^2 = \frac{MR^2}{2}\cos^2(\omega t) = \frac{MR^2}{4}(1 + \cos(2\omega t))$, $Q_{yy} = 2My^2 = \frac{MR^2}{2}\sin^2(\omega t) = \frac{MR^2}{4}(1 - \cos(2\omega t))$, and $Q_{xy} = Q_{yx} = 2Mxy = \frac{MR^2}{4}\sin(2\omega t)$.

(e) By taking the third derivatives in time, we have $\dddot{Q}_{xx} = \frac{MR^2}{4}(2\omega)^3\sin(2\omega t) = 2MR^2\omega^3\sin(2\omega t)$. Similarly, it is easy to get $\dddot{Q}_{yy} = -2MR^2\omega^3\sin(2\omega t) = -\dddot{Q}_{xx}$ and $\dddot{Q}_{xy} = -2MR^2\omega^3\cos(2\omega t)$. From these, we have $\sum_{i,j}\dddot{Q}_{ij}\dddot{Q}_{ij} = (\dddot{Q}_{xx})^2 + (\dddot{Q}_{yy})^2 + 2(\dddot{Q}_{xy})^2 = 8M^2R^4\omega^6$ and $\sum_i \dddot{Q}_{ii} = \dddot{Q}_{xx} + \dddot{Q}_{yy} = 0$. Inserting this to the expression for P and using the fact that $\omega = 2\pi/T = \sqrt{\frac{2GM}{R^3}}$, we obtain $P = \frac{64}{5}\frac{G^4M^5}{c^5R^5}$. The total mechanical energy is half of the gravitational energy by the virial theorem, $E = -\frac{GM^2}{2R}$. From $\frac{dE}{dt} = -P$ we get an equation $\frac{dR(t)}{dt} = -\frac{128}{5}\frac{G^3M^3}{c^5R(t)^3}$, which is integrable in time.

5. (a) From the parallel axis theorem, the moment of inertia about O is $I = I_{CM} + Ml^2$.

   (b) The torque about the point O is provided only by the gravitational force acting on the center of mass, and it is $Mgl\sin\theta$. Equating this with $I\ddot{\theta}$, we have $\ddot{\theta} = \frac{Mgl}{I}\sin\theta = \frac{Mgl}{I_{CM}+Ml^2}\sin\theta$.

   (c) The total mechanical energy is a sum of the kinetic energy and the gravitational potential energy. The former is $K = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}(I_{CM} + Ml^2)\dot{\theta}^2$, and the latter is $V = Mgl\cos\theta$. The total mechanical energy should be equal to its initial value at $t = 0$, which is Mgl. Therefore, we have $\frac{1}{2}(I_{CM} + Ml^2)\dot{\theta}^2 + Mgl\cos\theta = Mgl$, which gives us $\dot{\theta}^2 = \frac{2Mgl}{I_{CM}+Ml^2}(1 - \cos\theta)$.

   (d) From $x = l\sin\theta$, we have $\dot{x} = l\cos\theta\dot{\theta}$ and $\ddot{x} = -l\sin\theta\dot{\theta}^2 + l\cos\theta\ddot{\theta}$. From the Newton's second law of motion, $M\ddot{x}$ should be equal to the total horizontal force acting on the system, which in our case is the friction force F. This gives $F = Ml(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2)$. Using the expressions for $\ddot{\theta}$ and $\dot{\theta}^2$ from (b) and (c), we find $F = Mg\left(\frac{Ml^2}{I_{CM}+Ml^2}\right)(3\sin\theta\cos\theta - 2\sin\theta)$. Similarly, $M\ddot{y} = -Ml(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2)$ should be equal to the total vertical force on the system, which is $N - Mg$. This gives, after a short computation, $N = Mg - Ml(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2) = Mg\left(\frac{I_{CM}+Ml^2(3\cos^2\theta-2\cos\theta)}{I_{CM}+Ml^2}\right)$.

   (e) From the condition $N = 0$, we obtain $-3\cos^2\theta_0 + 2\cos\theta_0 = \frac{I_{CM}}{Ml^2}$. The function $-3x^2 + 2x$ defined in $x \in (0, 1)$ has a maximum value $\frac{1}{3}$ at $x = \frac{1}{3}$. Therefore, the above equation has a solution θ_0 if and only if $\frac{I_{CM}}{Ml^2} < \frac{1}{3}$.

BACK UP FOR PROBLEM 4

1. Let's define the quadrupole moments of the system Q_{ij} by $Q_{ij} = \sum_\alpha m_\alpha x_{\alpha i} x_{\alpha j}$ where α runs over the two stars, $m_\alpha = M$ is the mass, and $x_i = x, y, z$ are the Cartesian coordinates of the star from its center of mass. Writing the coordinates of the first star in time t as $x = \frac{R}{2}\cos(\omega t)$ and $y = \frac{R}{2}\sin(\omega t)$ $(z = 0)$ with $\omega = 2\pi/T$, write down an expression for Q_{xx}, Q_{yy} and $Q_{xy} = Q_{yx}$.

2. It is known that a binary system loses its energy by gravitational radiation with the rate given by

$$P = \frac{G}{5c^5}\left(\sum_{i,j=1}^{3} \dddot{Q}_{ij}\dddot{Q}_{ij} - \frac{1}{3}\left(\sum_{i=1}^{3}\dddot{Q}_{ii}\right)^2\right), \tag{0.2}$$

   where $\cdot \equiv \frac{d}{dt}$. Show that $P = \frac{64}{5}\frac{G^4M^5}{c^5R^5}$ for our system. Considering the total mechanical energy in terms of R, write down a time evolution equation for $R(t)$.