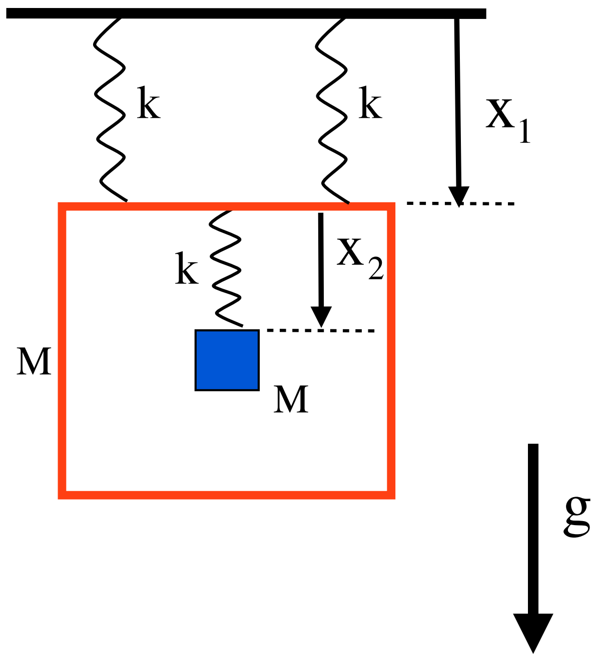
Classical Mechanics

1. Consider a box of mass M hanging vertically with two identical parallel springs of a spring constant k. Inside the box, there is another spring with the same spring constant k hanging from the ceiling of the box with a bob of mass M attached to its end. Let the acceleration of gravity be g. Let's denote the displacement of the box from the top by x_1, and the displacement of the bob of mass *from the ceiling of the box* by x_2.



(a) Find the equilibrium value of x_1 and x_2, denoted as x_1^{eq} and x_2^{eq} respectively.

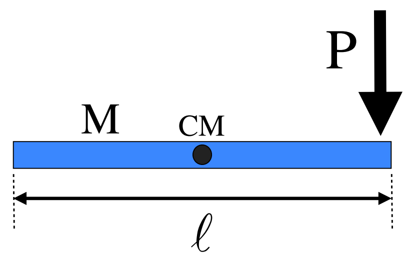
(b) Consider small displacements of x_1 and x_2 from their equilibrium values by writing $x_1 = x_1^{eq} + X_1$ and $x_2 = x_2^{eq} + X_2$. Write down the Lagrangian of the system in terms of X_1 and X_2, and show that there is no linear term in X_1 and X_2, that is, the Lagrangian is quadratic in X_i and $\dot{X}_i$ $(i = 1, 2)$.

(c) Obtain the coupled equations of motions for X_1 and X_2.

(d) Find the normal modes and the normal mode frequencies of the system. Give qualitative descriptions of each normal mode motion.

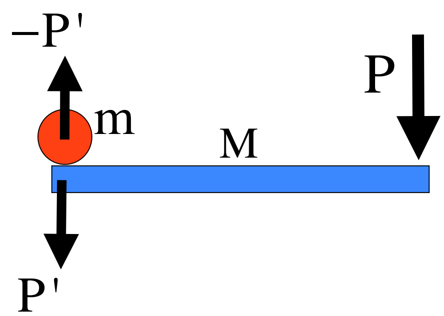
2. Consider a uniform thin rod of length ℓ with a mass M and a moment of inertia about the center of mass $I_{CM} = \frac{1}{12}M\ell^2$. Assume that it is initially at rest on a *frictionless* surface.

   (a) A sudden horizontal impulse of magnitude P is applied perpendicular to the rod at one end, as shown in the figure below (which is a top view). Find the instantaneous velocity of the center of mass and the instantaneous angular velocity about the center of mass immediately after the impulse.



   (b) Find the instantaneous velocity of the other end of the rod immediately after the impulse. Show that its direction is opposite to the direction of the impulse P.

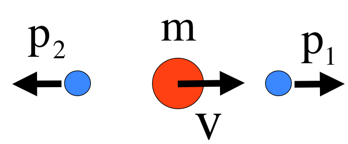
   (c) Consider a situation where the rod is initially at rest, and there is a separate mass m at the other end as shown below when a sudden impulse P is applied to one end. Find the velocity of the mass after the impulse. (Hint: Use the fact that the mass and the other end of the rod exchange a perpendicular impulse of magnitude P' such a way that the instantaneous velocities of the mass and the other end of the rod are the same immediately after the original impulse P).



   (d) Show or argue that the final velocity of the center of mass of the rod in the second situation (i.e. in part (c)) is greater than that of the first situation (i.e. in part (a)), while the angular velocity in the second is smaller than that of the first.

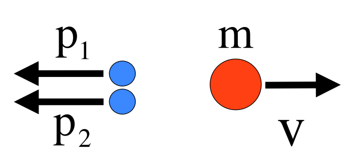
3. A heavy nucleus of mass M initially at rest in the lab frame undergoes a neutrinoless double β-decay to become a lighter nucleus of mass m while emitting two electrons. Let's assume that the rest mass of electrons is negligibly small, and we treat them as massless. We would like to find the maximum recoil velocity of the final nucleus in the lab frame. All parts of this problem should be solved using the special relativity.

   (a) Consider a final state where two electrons move in opposite directions and the final nucleus moves in the same direction of one of the electrons as shown in the figure. Let p_1 and p_2 be the *magnitudes* of the two electrons' momenta as shown in the figure. Write down the energy and momentum conservation equations in terms of p_1, p_2, M, m and v (you can also use the notation for $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$ where c is the speed of light).



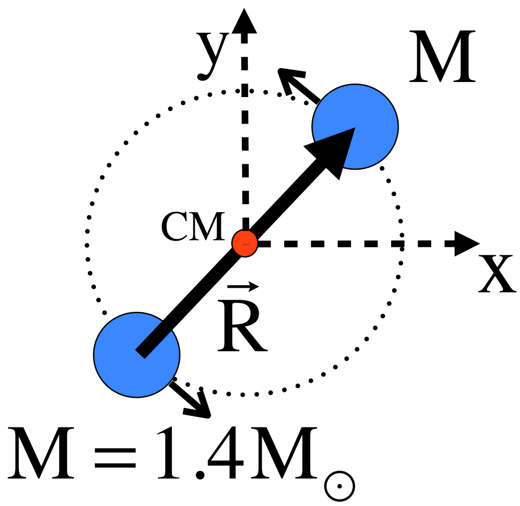
   (b) We can generally assume that $p_2 \geq p_1$ so that $v \geq 0$. Using the energy and momentum conservation equations, find an expression of p_1 in terms of M, m and v. Imposing the condition $p_1 \geq 0$, obtain the *maximum* possible value of v in terms of M and m.

   (c) Consider now a different final state where the two electrons move in a same direction while the final nucleus moves in the opposite direction with a velocity v as in the figure below. Write down the energy and momentum conservation equations in terms of p_1, p_2, M, m and v. Show that we can determine v in terms of M and m without knowing p_1 and p_2, and find its expression. Compare your result with the maximum velocity in part (c).



   (d) Give a qualitative argument why the velocity you find in part (c) is the maximum recoil velocity of the final nucleus in arbitrary final states of the electrons.

4. An astronomer observes a binary system of two neutron stars of equal mass M of about 1.4 times of the solar mass $M_\odot$. The two neutron stars are orbiting around the static center of mass in a circular orbit, and the distance between the stars is R. Let G be the Newton's gravitation constant. We assume that non-relativistic Newton's law of motion and gravity apply to this system.



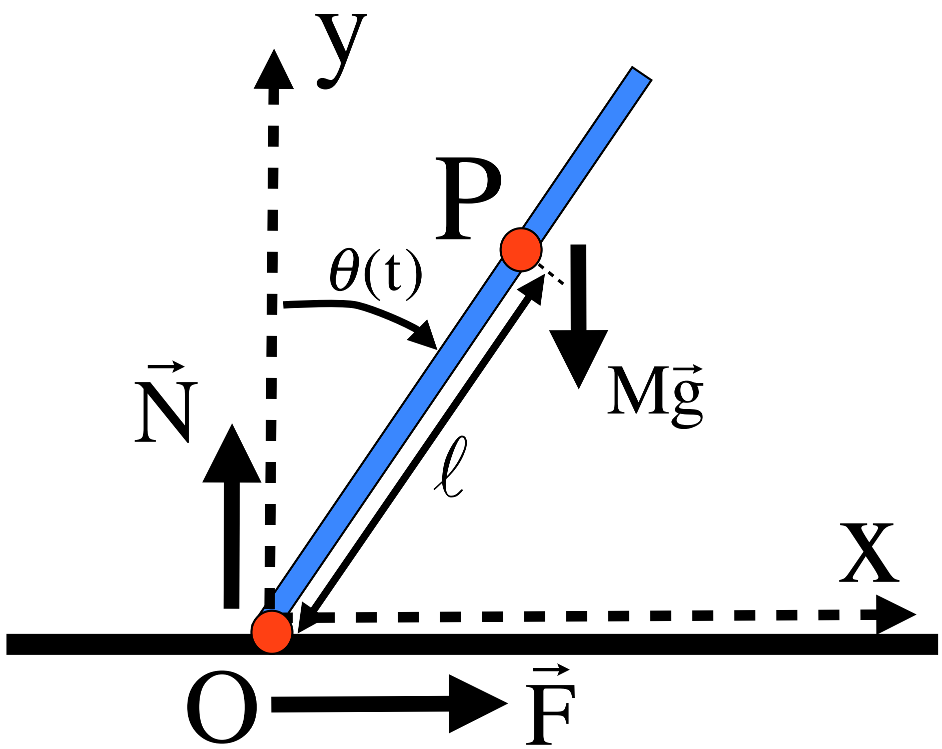
   (a) Find the reduced mass of the system, and show that the equation of motion for the distance vector from one star to the other, denoted by $\vec{R}$, is equal to the equation of motion of a test particle under a gravitational force produced by an object of mass $2M$.

   (b) Find the period of the motion T in terms of M, R and G.

   (c) If $R = 150$ km, estimate the frequency of the binary neutron star orbit in Hertz (Hz). Use the information that the distance of the Earth from the Sun is about $R_\odot = 1.5 \times 10^8$ km, one year is about 3.2×10^7 seconds, and $\sqrt{2.8} \approx 1.7$ (Do *not* use the value of G or the mass of the Sun).

   (d) It is known that the binary system loses its energy by gravitational radiation with the rate given by $P = \frac{64}{5}\frac{G^4 M^5}{c^5 R^5}$ where c is the speed of light. Considering the total mechanical energy in terms of R, write down a time evolution equation for $R(t)$.

5. A statue whose center of mass P is located at a distance ℓ from its toe (point O) starts to fall down from its vertical position as shown in the figure. The ground surface is rough enough to provide a necessary horizontal friction force for this motion to happen. It can also provide a vertical normal force if necessary. Let the moment of inertia about the center of mass be I_{CM}, and the total mass be M. The acceleration of gravity is g.



(a) Find the moment of inertia about the point O in terms of M, ℓ and I_{CM}.

(b) Let the angle the center of mass makes from the vertical line at time t be $\theta(t)$, and assume the initial conditions $\theta(0) = \dot{\theta}(0) = 0$. Find the equation of motion of $\theta(t)$, and express $\ddot{\theta}$ in terms of θ, M, ℓ, I_{CM} and g.

(c) Using the conservation of total mechanical energy or integrating the equation of motion, express $\dot{\theta}^2$ in terms of θ, M, ℓ, I_{CM} and g.

(d) The Cartesian coordinates of the center of mass from the point O is $x(t) = \ell \sin\theta(t)$ and $y(t) = \ell\cos\theta(t)$. Considering Newton's second law of motion for the center of mass, express the normal force $\vec{N}$ and the frictional force $\vec{F}$ provided by the ground surface in terms of θ, M, ℓ, I_{CM} and g.

(e) The toe of the statue leaves the ground surface at some angle $0 < \theta_0 < \frac{\pi}{2}$ when the normal force vanishes. Find an equation for θ_0, and show that this happens if and only if $I_{CM} < \frac{1}{3}M\ell^2$.