1.

One mole of monoatomic ideal gas initially at temperature T_0 expands from volume V_0 to $2V_0$. Calculate the work W performed by the gas and the heat Q absorbed by it in three cases:

(a) the expansion is isothermal;

(b) the expansion is isobaric;

(c) the expansion is adiabatic.

2.

A quantum system has only two possible energies 0 and $\Delta > 0$. While the ground state is unique, the state with energy Δ has degeneracy g.

(a) Find the free energy of the system as a function of temperature.

(b) Find the entropy of the system. What is the maximum value of the entropy?

(c) At what temperatures is the most probable value of the energy of the system equal to Δ? Explain the meaning of this answer when $g = 1$.

3.

A body of constant heat capacity C_p and initial temperature T_1 is placed in contact with an infinite reservoir at temperature T_2. Equilibrium is established at constant pressure.

(a) Find the change of the entropy of the body.

(b) Find the change of the entropy of the reservoir.

(c) Show that the total entropy increases whether $T_1 > T_2$ or vice versa.

4.

A paramagnetic system is thermally insulated and placed in a magnetic field H. The induced magnetization depends on the temperature and magnetic field as $M = aH/T$, while the heat capacity at constant H is given by $c_H = b/T^2$, where a and b are known constants.

(a) Using the differential of the free energy $dF = -SdT - MdH$, express the derivative $(\partial S/\partial H)_T$ in terms of $(\partial M/\partial T)_H$.

(b) Use the above result to calculate $(\partial T/\partial H)_S$.

(c) Detemine the final value T_1 of the temperature of the body given that the initial temperature was T_0 and the initial magnetic field H_0 and that the magnetic field was reduced to zero quasi-statically.

5.

Consider a model of a semiconductor consisting of N non-interacting electrons which can occupy either a bound state with energy $\varepsilon = -E_b$ or a free-particle continuum with energy $\varepsilon = \dfrac{p^2}{2m}$.

(a) Write the expression for the number N_b of electrons in the bound state at a given temperature T and chemical potential μ.

(b) Given chemical potential μ and temperature T write down the number N_f of free electrons assuming that the temperature is low enough that there are very few free electrons (non-degenerate limit).

(c) Using these results determine the value of the chemical potential μ and the number of free electrons N_f as a function of T, E_b and m in the limit when the temperature is low enough that $N_f \ll N$.

(1) Equation of state : $pV = RT$, $C_V = \frac{3}{2}RT$

(a) $T = \text{const}$

$$W = \int_{V_0}^{2V_0} p\,dV = RT_0 \int_{V_0}^{2V_0} \frac{dV}{V} = RT_0 \ln 2$$

since $\Delta U = 0$:

$$Q = W = RT_0 \ln 2$$

(b) $p = \text{const}$

$$W = \int_{V_0}^{2V_0} p\,dV = pV_0 = RT_0$$

$$\Delta U = C_V \Delta T = \frac{3}{2} R \Delta T = \frac{3}{2} p \Delta V = \frac{3}{2} p V_0 = \frac{3}{2} R T_0$$

Thus:

$$Q = \Delta U + W = \frac{5}{2} R T_0$$

(c) $Q = 0$

$$dS = \frac{dU}{T} + \frac{p}{T} dV = \frac{3}{2} R \frac{dT}{T} + \frac{R}{V} dV = d(R \ln V T^{3/2})$$

$$dS = 0 \Rightarrow T = (T_0 V_0^{2/3}) V^{-2/3}$$

$$W = \int_{V_0}^{2V_0} p\,dV = R \int_{V_0}^{2V_0} \frac{T}{V} dV = R(T_0 V_0^{2/3}) \int_{V_0}^{2V_0} V^{-\frac{5}{3}} dV$$

$$= RT_0 \frac{3}{2}\left(1 - 2^{-\frac{2}{3}}\right) \simeq RT_0 \cdot 0.555$$

(Can start from $pV^{5/3} = \text{const} = p_0 V_0^{5/3} = RT_0 V_0^{2/3}$)

(2)

(a) $F = -T\ln Z = -T\ln\left(1 + ge^{-\Delta/T}\right)$

(b) $S = -\frac{\partial F}{\partial T} = \ln\left(1+ge^{-\Delta/T}\right) + \frac{\Delta}{T}\frac{g}{e^{\Delta/T}+g}$

When $T\to\infty$ $S \to \ln(1+g)$ – maximum value

(c) When $p(\Delta) > p(0)$,

i.e. $ge^{-\Delta/T} > 1$

or $T > \Delta/\ln g$

when $g=1$, $T>\infty$ means most probable energy is always 0.

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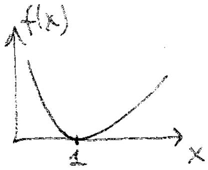
(3) (a) $\Delta S_1 = \int_{T_1}^{T_2} \frac{C_p\,dT}{T} = C_p \ln\frac{T_2}{T_1}$

(b) $\Delta S_2 = \frac{\Delta Q}{T_2} = \frac{C_p(T_2-T_1)}{T_2}$

(c) $\Delta S = \Delta S_1 + \Delta S_2 = C_p\left(\frac{T_1}{T_2} - 1 - \ln\frac{T_1}{T_2}\right) = C_p\,(x-1-\ln x)$, where $x = T_1/T_2$

$f(x) = x-1-\ln x > 0$ for $x>0$, except $f(1)=0$



(4)

(a)

$$dF = -S\,dT - M\,dH$$

Using Maxwell relation:

$$\left(\frac{\partial S}{\partial H}\right)_T = \left(\frac{\partial M}{\partial T}\right)_H$$

(b) $$\left(\frac{\partial T}{\partial H}\right)_S = -\underbrace{\left(\frac{\partial T}{\partial S}\right)_H}_{=T/C_H}\left(\frac{\partial S}{\partial H}\right)_T = -\frac{T}{C_H}\left(\frac{\partial M}{\partial T}\right)_H = \frac{aTH}{b}$$

(c) Integrating above equation:

$$\ln T - \frac{a}{2b}H^2 = \text{const}$$

Using initial condition $T=T_0$ when $H=H_0$:

$$T = T_0\, e^{\frac{a}{2b}(H^2 - H_0^2)}$$

Thus $T = T_0\, e^{-\frac{a}{2b}H_0^2}$ when $H=0$

(5)

(a) Fraction of electrons in the bound state:

$$\frac{N_b}{N} = \frac{1}{e^{-(E_b+\mu)/T}+1}$$

(b) Number of free electrons (assuming $e^{\mu/T} \ll 1$)

$$N_f = 2\int \frac{d^3p}{(2\pi)^3}\, e^{(\mu-\frac{p^2}{2m})/T} = 2e^{\mu/T}\cdot\left(\frac{mT}{2\pi}\right)^{3/2} \equiv N_0 e^{\mu/T}$$

$\int dx\, e^{-x^2} = \sqrt{\pi}$

$N_0 \equiv 2\left(\frac{mT}{2\pi}\right)^{3/2}$

$x \equiv e^{-\mu/T}$

(c) Since $N_b + N_f = N$

$$\frac{N}{x e^{-E_b/T}+1} + \frac{N_0}{x} - N = 0$$

$$\cancel{Nx} + N_0(xe^{-E_b/T}+1) - N(x^2 e^{-E_b/T} + \cancel{x}) = 0$$

$$x^2 - \frac{N_0}{N}x - \frac{N_0}{N}e^{+E_b/T} = 0$$

$$x = \frac{N_0}{2N} + \sqrt{\left(\frac{N_0}{2N}\right)^2 + \frac{N_0}{N}e^{E_b/T}} \qquad (*)$$

$\mu = -T\ln x = \ldots$

In the low T limit $N_f \ll N$, $N_b \approx N$ and one can approximate:

$$N_b \approx N(1 - xe^{-E_b/T})$$

$$N(\cancel{1} - xe^{-E_b/T}) + \frac{N_0}{x} - \cancel{N} = 0$$

$$x = \left(\frac{N_0}{N}e^{E_b/T}\right)^{1/2} \quad \left(\text{one can also take } e^{E_b/T} \gg \frac{N_0}{N} \text{ in } (*)\right)$$

or $\mu = -T\ln x \approx -\frac{E_b}{2} + \frac{T}{2}\ln\frac{N}{N_0}$

$$N_f = \frac{N_0}{x} = \sqrt{N_0 N}\, e^{-\frac{E_b}{2T}}$$