University of Illinois at Chicago
Department of Physics

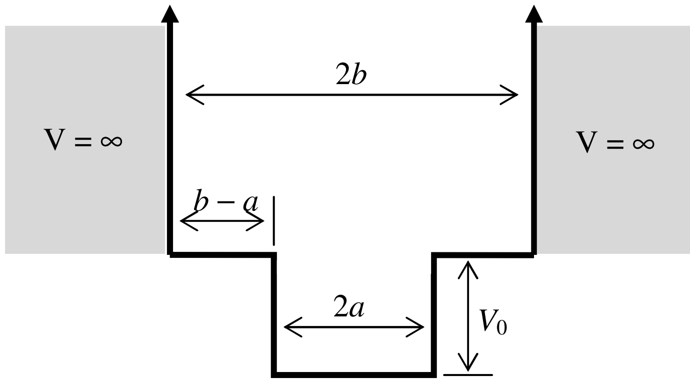
Quantum Mechanics
Qualifying Examination

January 3, 2017
9.00am – 12.00 pm

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted towards the exam's total score.

Various equations, constants, etc. are provided on the last page of the exam.

1. A particle of mass m is confined to a finite square well of depth V_0 and width $2a$ that is symmetrically enclosed by an infinite square well of width $2b$ ($b > a$).



a) If the ground state is bound to the finite square well (i.e., has an energy below the bottom of the infinite square well), show that this requires

$$\tanh[\chi(b-a)] = \frac{\chi}{k\tan(ka)} \; ,$$

where $k = \sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}}$, $\chi = \sqrt{\frac{2m|E|}{\hbar^2}}$, E is the confinement energy in the finite square well, and $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

b) Verify that the result of part (a) reduces to the expected form in the limits when (i) $b \to a$ and (ii) $b \to \infty$.

c) Find the algebraic relation, similar to that in part (a), for the first excited state if it is also bound to the finite square well.

2. A particle in a spherically symmetric potential is in a state described by the wave packet

$$\psi(x, y, z) = C\left(xy + yz + z^2\right)\exp\left(-\alpha r^2\right) .$$

a) What are the possible results of a measurement with $\hat{L}^2$ and with what probabilities would they occur?

b) What are the possible results of a measurement with $\hat{L}_z$ and with what probabilities would they occur?

3. In an orthonormal basis set $|1\rangle$, $|2\rangle$, $|3\rangle$, the Hamiltonian of a quantum system is represented by the following:

$$\hat{H}_0 = \alpha\{|1\rangle\langle 1| + |2\rangle\langle 2|\} + i\{|1\rangle\langle 2| - |2\rangle\langle 1|\} + \beta|3\rangle\langle 3| .$$

a) What are the energy eigenvalues of the system and what are their normalized eigenvectors in terms of the $|1\rangle$, $|2\rangle$, $|3\rangle$ basis?

b) The system is now perturbed by $\hat{H}_1 = \varepsilon\{|1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|\}$ where ε is small. Using non-degenerate perturbation theory, find the modified energies of the three quantum states?

c) Under what conditions is non-degenerate perturbation theory valid in this case?

4. Consider a system of spin ½ in the S_z-representation, where

$$\hat{S}_i = \tfrac{1}{2}\hbar\sigma_i \quad (i = x, y, z) \text{ with } \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) What are the eigenvalues and eigenvectors of the operator $\hat{O} = \hat{S}_x + \hat{S}_y$?

b) Suppose a measurement of this operator is made, and the system is found to be in a state corresponding to the larger eigenvalue. What is the probability that a measurement of $\hat{S}_z$ yields $+\hbar/2$?

c) What is the 2×2 unitary transformation matrix U from the S_z-representation into the (S_x+S_y)-representation? Check your answer by transforming the eigenvectors obtained in part (a).

d) Verify that $U^\dagger U = 1$, the identity matrix.

e) Evaluate $\hat{O}$ in the (S_x+S_y)-representation.

5. The central potential that best describes a quark-antiquark system is $V(r) = Cr$, where C represents the strength of the attractive potential.

a) Using a suitably parameterized version of the ground state hydrogenic wave function, $R_{10}(r)$, for $Z = 1$, prove that energy of the ground state may be approximated by

$$E_1 = \frac{3}{2}\left(\frac{9\hbar^2 C^2}{4\mu}\right)^{\frac{1}{3}},$$

where μ is the reduced mass.

b) Similarly, using $R_{20}(r)$, for $Z = 1$, find the energy of the first excited state.

c) If the excitation spectrum of a quark-antiquark system reveals that the energy difference between the excitation of the 1s and 2s states is $\left(1-\frac{1}{\sqrt[3]{4}}\right)mc^2$, find both the coefficient C describing the strength of the central potential and the excitation energy of the 1s state (i.e., E_{1s}) in terms of quark mass m, the reduced Planck's constant $\hbar$, the speed of light c, and numeric factors.

Equation Sheet

$$x = r\cos\phi\sin\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\theta$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}} \qquad Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{1\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\, e^{\pm i\phi}\sin\theta$$

$$Y_{20}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right) \qquad Y_{2\pm1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}\, e^{\pm i\phi}\sin\theta\cos\theta$$

$$Y_{2\pm2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\, e^{\pm 2i\phi}\sin^2\theta$$

$$E_n = -\left[\frac{\mu}{2\hbar^2}\left(\frac{Ze^2}{4\pi\varepsilon_0}\right)^2\right]\frac{1}{n^2} = -13.6\frac{Z}{n^2}\text{ eV} \qquad a_0 = \frac{4\pi\varepsilon_0\hbar^2}{\mu e^2} \qquad \frac{1}{\mu} = \frac{1}{m_e} + \frac{1}{M_{nucleus}}$$

$$R_{10}(r) = 2\left(\frac{Z}{a_0}\right)^{\frac{3}{2}}\exp\left[-\frac{Zr}{a_0}\right] \qquad R_{20}(r) = 2\left(\frac{Z}{2a_0}\right)^{\frac{3}{2}}\left(1 - \frac{Zr}{2a_0}\right)\exp\left[-\frac{Zr}{2a_0}\right]$$

$$R_{21}(r) = \frac{1}{\sqrt{3}}\left(\frac{Z}{2a_0}\right)^{\frac{3}{2}}\frac{Zr}{a_0}\exp\left[-\frac{Zr}{2a_0}\right]$$

$$\int_0^\infty dx\, x^m \exp\left(-ax^2\right) = \frac{\Gamma\left(\frac{m+1}{2}\right)}{2a^{(m+1)/2}} \;\; ; \;\; \Gamma(n+1) = n\Gamma(n),\ \Gamma(n+1) = n!,\ \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

so that $\displaystyle\int_0^\infty dx\, x^{2n}\exp\left(-\lambda^2x^2\right) = \frac{1.3.5\ldots(2n+1)\sqrt{\pi}}{2^n\lambda^{2n+1}}$

$$\int_0^\infty dx\, x^n e^{-\lambda x} = \frac{n!}{\lambda^{n+1}}$$