Question 1:

a/ Since both coils have the same current flow direction, the increased magn. flux through one coil will increase the flux in the other.

Therefore: $M > 0$

b/ Since the flow of the current in the lower coils is now flipped, we have to change the sign of M.

$\Rightarrow \quad M < 0$

c) From the figure (c), we see:

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = -(L_1 + L_2 + M + M)\frac{dI}{dt}$$

$$= -L'\frac{dI}{dt}$$

$$\Rightarrow \boxed{L' = L_1 + L_2 + 2M}$$

d) Now: $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = -(L_1 - M + L_2 - M)\dfrac{dI}{dt}$

$$= -(L_1 + L_2 - 2M)\frac{dI}{dt}$$

$$\Rightarrow \boxed{L'' = L_1 + L_2 - 2M}$$

Since $M > 0$, $L' > L''$.

e) To keep self-inductance positive:

$$L_1 + L_2 - 2M > 0$$

and $L_1 + L_2 + 2M > 0$

and $L_1, L_2 > 0$

So: $(L_1 + L_2 \pm 2M) > 0$

or $(\sqrt{L_1} \pm \sqrt{L_2})^2 > 0$ if $\boxed{\sqrt{L_1 L_2} \geq M}$

Question 2:

a)
$$Q = \int_0^\infty \rho(r)\, dV = \int_0^{R/2} \alpha \, dV + \int_{R/2}^{R} 2\alpha\left(1-\frac{r}{R}\right) dV$$

$$dV = r^2 dr \; d(\sin\theta)\; d\phi$$

$$Q = \alpha \int_0^{R/2} r^2 4\pi \, dr + 2\alpha \int_{R/2}^{R} \left(1-\frac{r}{R}\right) 4\pi r^2 dr$$

$$= \alpha \frac{4\pi}{3} \frac{R^3}{2^3} + 2\alpha \cdot 4\pi \left[\frac{1}{3} r^3 - \frac{1}{4}\frac{r^4}{R}\right]\Bigg|_{R/2}^{R}$$

$$= \frac{\alpha\pi R^3}{6} + 8\pi\alpha\left[\frac{1}{3}R^3 - \frac{1}{24}R^3 - \frac{1}{4}R^3 + \frac{1}{64}R^3\right]$$

$$Q = \frac{5}{8}\pi\alpha R^3 \implies \boxed{\alpha = \frac{8Q}{5\pi R^3}}$$

b) For $r \leq \frac{R}{2}$:

Use $\vec{\nabla}\vec{E} = \rho/\varepsilon_0 \Rightarrow \oint \vec{E}\, d\vec{a} = \frac{1}{\varepsilon_0}\int \rho \, dV$ (Gauss's Law)

$$\Rightarrow E \cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\alpha \int_0^r 4\pi r^2 dr = \frac{4\pi\alpha r^3}{3\varepsilon_0} = \frac{4\pi}{3\varepsilon_0}\frac{8Q}{5\pi R^3} r^3$$

$$\Rightarrow \boxed{\vec{E} = \frac{8Q}{15\pi\varepsilon_0 R^3}\vec{r}}$$

For $R/2 \le r \le R$: Same approach:

$$E \cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\left[\alpha \int_0^{R/2} 4\pi r^2 dr + 4\pi \cdot 2\alpha \int_{R/2}^{r} \left(1 - \frac{r'}{R}\right) r'^2 dr'\right]$$

$$= \frac{\pi \alpha R^3}{6\varepsilon_0} + \frac{8\pi\alpha}{\varepsilon_0}\left[\frac{1}{3} r'^3 - \frac{1}{4}\frac{r'^4}{R}\right]\Bigg|_{R/2}^{r}$$

$$= \frac{\pi\alpha R^3}{6\varepsilon_0} + \frac{8\pi\alpha}{\varepsilon_0}\left[\frac{1}{3} r^3 - \frac{1}{4}\frac{r^4}{R} - \frac{R^3}{24} + \frac{R^3}{64}\right]$$

$$= -\frac{\pi\alpha R^3}{24\varepsilon_0} + \frac{8\pi\alpha}{3\varepsilon_0} r^3 - \frac{2\pi\alpha r^4}{\varepsilon_0 R}$$

$$\Rightarrow \quad E = \frac{2\alpha}{3\varepsilon_0} r - \frac{\alpha}{2\varepsilon_0 R} r^2 - \frac{\alpha R^3}{96\varepsilon_0}\frac{1}{r^2}$$

or $$\boxed{\vec{E} = \left[\frac{16Q}{15\pi\varepsilon_0}\frac{r}{R^3} - \frac{4\ Q}{5\pi\varepsilon_0}\frac{r^2}{R^4} - \frac{Q}{60\pi\varepsilon_0}\frac{1}{r^2}\right]\hat{r}}$$

For $r \ge R$: $\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}$

c) Use $r \le R/2$ and a restoring force of

$F_r = -eE$

From (b), we find $E = \frac{8Q}{15\pi\varepsilon_0 R^3} r$ for $0 < r \le \frac{R}{2}$

$\Rightarrow F_r = -\frac{8Qe}{15\pi\varepsilon_0 R^3} r \Rightarrow F_r \sim r$

Since the restoring force depends on r, we have simple harmonic oscillation! Similar to spring $(F = -kx)$

Use spring analogy: $F = -kx$ where $k = \frac{8Qe}{15\pi\varepsilon_0 R^3}$

Frequency for spring motion: $T = \frac{2\pi}{\omega}$ where $\omega = \sqrt{\frac{k}{m}}$

$$\Rightarrow \boxed{T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{15\pi\varepsilon_0 m R^3}{8Qe}}}$$

Question 3:

a) Using electric dipoles and the dipole moment, $\vec{p}(t)$

$$\vec{p}(t) = eR\,(\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)) + eR(\hat{x}\cos(\omega t+\alpha) + \hat{y}\sin(\omega t+\alpha))$$

or $$\vec{p}(t) = eR\,\mathrm{Re}\left[(\hat{x}+i\hat{y})e^{-i\omega t} + (\hat{x}+i\hat{y})e^{-i(\omega t+\alpha)}\right]$$

$$= eR\,\mathrm{Re}\left[(\hat{x}+i\hat{y})e^{-i\omega t}(1+e^{-i\alpha})\right]$$

The electric dipole Radiation in the radiation zone:

$$\vec{E} = \frac{\mu_0}{4\pi r}\left[\hat{r}\times(\hat{r}\times\ddot{\vec{p}})\right] \text{ at } t_0 = t - r/c$$

Therefore: $$\vec{E} = \frac{eR\mu_0}{4\pi r}(-\omega^2)e^{-i\omega(t-r/c)}(1+e^{-i\alpha}) \cdot \left[\hat{r}\times(\hat{r}\times(\hat{x}+i\hat{y}))\right]$$

$$\Leftrightarrow \vec{E} = -\frac{eR\mu_0\omega^2}{4\pi r}e^{i(k\vec{r}-\omega t)}(1+e^{-i\alpha})\left[\hat{r}\times(\hat{r}\times\hat{x}) + i\,\hat{r}\times(\hat{r}\times\hat{y})\right]$$

use $k = \frac{\omega}{c}$

Magnetic field: $\vec{B} = \hat{r}\times\vec{E}$

b) $\frac{dP}{d\Omega} = \frac{1}{\mu_0 c} |E|^2 r^2 = \frac{1}{\mu_0 c} |E^* \cdot E| r^2$

$= \left(\frac{eR\mu_0\omega^2}{4\pi r}\right)^2 \frac{r^2}{\mu_0 c} (1+e^{i\alpha})(1+e^{-i\alpha})\left[(\hat{r}\times(\hat{r}\times\hat{x}))^2 + (\hat{r}\times(\hat{r}\times\hat{y}))^2\right]$

$= \frac{e^2 R^2 \mu_0 \omega^4}{16\pi c} (2+2\cos\alpha) \cdot \left[(\hat{r}\times\hat{x})^2 + (\hat{r}\times\hat{y})^2\right]$

$\hat{r}\times\hat{x} = \sin\theta_{\hat{r},\hat{x}} \Rightarrow (\hat{r}\times\hat{x})^2 = \sin^2\theta_{\hat{r},\hat{x}} = 1-(\hat{r}\cdot\hat{x})^2$

$r\times\hat{y} = \sin\theta_{r,\hat{y}} \rightarrow (\hat{r}\times\hat{y})^2 = \sin^2\theta_{\hat{r},\hat{y}} = 1-(\hat{r}\cdot\hat{y})^2$

$\Rightarrow \frac{dP}{d\Omega} = \frac{e^2R^2\mu_0\omega^4}{16\pi c} 4\cos^2\left(\frac{\alpha}{2}\right)\left[(1-\sin^2\theta\cos^2\phi)+(1-\sin^2\theta\sin^2\phi)\right]$

$$\boxed{\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi c} e^2R^2\omega^4 \cos^2\left(\frac{\alpha}{2}\right) \cdot (1+\cos^2\theta)}$$

c) For $\alpha = 0$: This can be treated as a single particle with charge $2e$ spinning around the origin and

$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi c} e^2 R\omega^4 (1+\cos^2\theta)$

For $\alpha = \pi$: $\frac{dP}{d\Omega} = 0$.

Question 4: (a) The boundary conditions for this potential are:

i. For $r \to \infty$: $V = 0$.
ii. For $r \to 0$: V is finite.
iii. $V_{inside} = V_{outside}$ for $r = R$.
iv. $\partial_r V_{inside} - \partial_r V_{outside} = \frac{\sigma}{\varepsilon_0}$ at $r = R$.

First, let's set $k = \frac{\sigma_0}{2}$. To solve this problem, you need to find the solution to the Laplace Equation in spherical coordinates so that the potential V can be written in this case as

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta).$$

Using Boundary Condition (i), we can find that $A_l = 0$ for $V_{outside}$. Similarly, using Boundary Condition (ii), we find that $B_l = 0$ for V_{inside}. Now, using Boundary Condition #3, we can write:

$$A_l\ R^l\ P_l(\cos\theta) = B_l \frac{1}{R^{l+1}}\ P_l(\cos\theta)$$

Since the Legendre Polynomials are orthogonal, we can rewrite this as:

$$B_l = A_l\ R^{2l+1}$$

Using the final Boundary Condition:

$$\begin{aligned}
\partial_r V_{inside} - \partial_r V_{outside} &= \frac{\sigma}{\varepsilon_0} \text{ at } r = R \\
\Leftrightarrow \sum_{l=0}^{\infty} \left(-l\ A_l R^{l-1} - (l+1)\frac{B_l}{R^{l+2}}\right) P_l(\cos\theta) \\
= \sum_{l=0}^{\infty} \left(-l\ A_l R^{l-1} - (l+1)\frac{A_l\ R^{2l+1}}{R^{l+2}}\right) P_l(\cos\theta) \\
\sum_{l=0}^{\infty} \left(-(2l+1) A_l R^{l-1}\right) P_l(\cos\theta) &= \frac{\sigma_0}{2\varepsilon_0}\left[3(\cos\theta)^2 - 1\right].
\end{aligned}$$

Now, remember that the surface charge density can be written in terms of the Legendre Polynomials as $\sigma = \sigma_0\, P_2(\cos\theta)$.

To solve this equation, we need to make use of the orthogonality of the Legendre Polynomials:

$$\sum_{l=0}^{\infty} \left(-\left(2l+1\right) A_l R^{l-1}\right) \int_0^{\pi} P_l(\cos\theta)\ P_m(\cos\theta)\ \sin\theta\, d\theta = \frac{\sigma_0}{\varepsilon_0} \left(\int_0^{\pi} P_2(\cos\theta)\ P_m(\cos\theta)\ \sin\theta\, d\theta\right).$$

This means that for every l, m other than $l = m = 2$ the integral is zero. And for $l = 2$:

$$-5A_2 R = \frac{\sigma_0}{\varepsilon_0}$$
$$\Leftrightarrow A_2 = -\frac{\sigma_0}{5\varepsilon_0 R} \text{ and}$$
$$B_2 = A_2 R^5 = -\frac{\sigma_0}{5\varepsilon_0} R^4.$$

So, we can write the potential inside the sphere now as:

$$V_{inside} = A_2 r^2 P_2(\cos\theta) = -\frac{\sigma_0 r^2}{10\varepsilon_0 R} \left[3\left(\cos\theta\right)^2 - 1\right].$$

Using $Q = 4\pi R^2\, \sigma_0$, we can now write the potential as:

$$\boxed{V_{inside} = -\frac{Q r^2}{40\pi\varepsilon_0 R^3} \left[3\left(\cos\theta\right)^2 - 1\right]}$$

The potential outside the sphere can now be written as:

$$V_{outside} = \frac{B_2}{r^3} P_2(\cos\theta) = -\frac{\sigma_0}{5\varepsilon_0} R^4 \frac{1}{r^3} \left[\frac{3}{2}\left(\cos\theta\right)^2 - 1\right].$$

Using $Q = 4\pi R^2\, \sigma_0$, we can now write the potential as:

$$\boxed{V_{outside} = -\frac{Q}{40\pi\varepsilon_0} R^2 \frac{1}{r^3} \left[3\left(\cos\theta\right)^2 - 1\right]}$$

b) $$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}')}{\scriptr} \, da',$$

where $\mathbf{K} = \sigma \mathbf{v}$, $\scriptr = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$, and $da' = R^2 \sin\theta' \, d\theta' \, d\phi'$. Now the velocity of a point $\mathbf{r}'$ in a rotating rigid body is given by $\boldsymbol{\omega} \times \mathbf{r}'$; in this case,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega \sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$= R\omega[-(\cos\psi\sin\theta'\sin\phi')\,\hat{\mathbf{x}} + (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\,\hat{\mathbf{y}} + (\sin\psi\sin\theta'\sin\phi')\,\hat{\mathbf{z}}]$.

Notice that each of these terms, save one, involves either $\sin\phi'$ or $\cos\phi'$. Since

$$\int_0^{2\pi} \sin\phi' \, d\phi' = \int_0^{2\pi} \cos\phi' \, d\phi' = 0,$$

such terms contribute nothing. There remains

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 R^3 \sigma\omega \sin\psi}{2} \left(\int_0^{\pi} \frac{\cos\theta' \sin\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}} \, d\theta' \right) \hat{\mathbf{y}}.$$

Letting $u \equiv \cos\theta'$, the integral becomes

$$\int_{-1}^{+1} \frac{u}{\sqrt{R^2 + r^2 - 2Rru}} \, du = -\frac{(R^2 + r^2 + Rru)}{3R^2r^2} \sqrt{R^2 + r^2 - 2Rru} \, \Bigg|_{-1}^{+1}$$

$$= -\frac{1}{3R^2r^2} \left[(R^2 + r^2 + Rr)|R - r| - (R^2 + r^2 - Rr)(R + r) \right].$$

If the point $\mathbf{r}$ lies *inside* the sphere, then $R > r$, and this expression reduces to $(2r/3R^2)$; if $\mathbf{r}$ lies *outside* the sphere, so that $R < r$, it reduces to $(2R/3r^2)$. Noting that $(\boldsymbol{\omega} \times \mathbf{r}) = -\omega r \sin\psi\,\hat{\mathbf{y}}$, we have, finally,

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 R \sigma}{3} (\boldsymbol{\omega} \times \mathbf{r}), & \text{for points } \textit{inside} \text{ the sphere,} \\[2ex] \dfrac{\mu_0 R^4 \sigma}{3r^3} (\boldsymbol{\omega} \times \mathbf{r}), & \text{for points } \textit{outside} \text{ the sphere.} \end{cases}$$

Having evaluated the integral, I revert to the "natural" coordinates of Fig. 5.45, in which $\boldsymbol{\omega}$ coincides with the z axis and the point $\mathbf{r}$ is at (r, θ, ϕ):

$$\mathbf{A}(r, \theta, \phi) = \begin{cases} \dfrac{\mu_0 R \omega\sigma}{3} r \sin\theta \, \hat{\boldsymbol{\phi}}, & (r \le R), \\[2ex] \dfrac{\mu_0 R^4 \omega\sigma}{3} \dfrac{\sin\theta}{r^2} \, \hat{\boldsymbol{\phi}}, & (r \ge R). \end{cases}$$

c)

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{2}{3}\mu_0\sigma R\omega\,\hat{\mathbf{z}} = \frac{2}{3}\mu_0\sigma R\boldsymbol{\omega}.$$

Question 5:

a) Use $\vec{\nabla} \times \vec{E} = -\partial_t B$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \partial_t \vec{D}$$

for $\vec{J}_f = 0$ and $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ and $\vec{H} = \frac{1}{\mu_0} \vec{B} - \underbrace{\vec{M}}_{=0}$

$$\vec{\nabla} \times \vec{H} = \varepsilon_0 \partial_t E + \partial_t \vec{P}$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = -\partial_t \left(\vec{\nabla} \times \vec{B} \right) = -\partial_t \left(\mu_0 \vec{\nabla} \times \vec{H} \right)$$

$$\vec{\nabla} \underbrace{(\vec{\nabla} \cdot \vec{E})}_{0} - \nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \partial_t^2 \vec{E} - \mu_0 \partial_t^2 \vec{P}$$

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \partial_t^2 E - \mu_0 \partial_t^2 \vec{P}$$

b) Using plane wave solution $\left(e^{i(kz - \omega t)} \right)$ and $\vec{P} = \varepsilon_0 \overleftrightarrow{\chi} \vec{E}$.

For $\hat{x}$: $-E_x k^2 = -\mu_0 \varepsilon_0 E_x \omega^2 - \mu_0 \varepsilon_0 \omega^2 (\chi_{11} E_x + i \chi_{12} E_y)$

y: $-k^2 E_y + \mu_0 \varepsilon_0 \omega^2 E_y = -\mu_0 \varepsilon_0 \omega^2 (-i \chi_{12} E_x + \chi_{11} E_y)$

z: $\nabla^2 E_z + \mu_0 \varepsilon_0 \omega^2 E_z = -\mu_0 \varepsilon_0 \omega^2 \chi_{33} E_z$

Using $\vec{\nabla} \cdot \vec{E} = 0 \Rightarrow \partial_x E_x + \partial_y E_y + \partial_z E_z = 0$

For wave travelling in z-direction, E_x and E_y are indep. of x and y $\Rightarrow \partial_x E_x = \partial_y E_y = 0$

$$\Leftrightarrow \partial_z E_z = 0$$

$$\Rightarrow E_z = \text{const} \equiv 0$$

Wave is purely transverse!

c) The equations to x and y have non-trivial solutions, if

$$\begin{vmatrix} -k^2 + \mu_0 \varepsilon_0 \omega^2 (1+\chi_{11}) & i\mu_0 \varepsilon_0 \omega^2 \chi_{12} \\ -i\mu_0 \varepsilon_0 \omega^2 \chi_{12} & -k^2 + \mu_0 \varepsilon_0 \omega^2 (1+\chi_{11}) \end{vmatrix} = 0$$

$$\Rightarrow \boxed{k = \sqrt{\mu_0 \varepsilon_0}\, \omega \sqrt{1 + \chi_{11} \pm \chi_{12}}}$$

d) Substituting k_R and k_L into the eq. for x and y above

$$x: \quad -E_x \mu_0 \varepsilon_0 \omega^2 (1+\chi_{11}+\chi_{12}) + \mu_0 \varepsilon_0 \omega^2 E_x = -\mu_0 \varepsilon_0 \omega^2 (\chi_{11} E_x + i\chi_{12} E_y)$$

$$y: \quad -E_y \mu_0 \varepsilon_0 \omega^2 (1+\chi_{11}+\chi_{12}) + \mu_0 \varepsilon_0 \omega^2 E_y = -\mu_0 \varepsilon_0 \omega^2 (-i\chi_{12} E_x + \chi_{11} E_y)$$

$$\Rightarrow -(\chi_{11}+\chi_{12}) E_x = -\mu_0 \varepsilon_0 \omega^2 \chi_{11} E_x - i\mu_0 \varepsilon_0 \omega^2 \chi_{12} E_y$$

$$-(\chi_{11}+\chi_{12}) E_y = +\mu_0 \varepsilon_0 \omega^2 i\chi_{12} E_x - \mu_0 \varepsilon_0 \omega^2 \chi_{11} E_y$$

$$\Rightarrow \boxed{E_x = i E_y}$$

Same for $k_L \Rightarrow \boxed{E_x = -iE_y}$ $\Rightarrow$ circular pol. light.

e) $n = \frac{c}{v}$ $\quad v = \frac{\omega}{k} \Rightarrow n = \frac{kc}{\omega}$

Here: $n_R = \frac{c}{\omega} k_R$ and $n_L = \frac{c}{\omega} k_L$

$\Rightarrow n_R - n_L = \frac{c}{\omega}(k_R - k_L)$ $\quad$ using $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

$$\boxed{n_R - n_L = \sqrt{1+\chi_{11}+\chi_{12}} - \sqrt{1+\chi_{11}-\chi_{12}}}$$