University of Illinois at Chicago
Department of Physics

Electromagnetism
Qualifying Examination

January 4, 2017

9.00 am - 12.00 pm

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted towards the exam's total score.

Various equations, constants, etc. are provided on the last page of the exam.

1. Self-Inductance

In part (a) of the figure below, you see two coils with self-inductance L_1 and L_2. In the relative position shown, their mutual inductance is M. The direction of positive current and the direction of the electromotive force are shown for each coil by the arrow. The equations relating the currents and the electromotive forces are given by:

$$\mathcal{E}_1 = -L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt} \tag{1}$$

and

$$\mathcal{E}_2 = -L_2\frac{dI_2}{dt} - M\frac{dI_1}{dt} \tag{2}$$

(a) Determine the sign of M in the equations above.

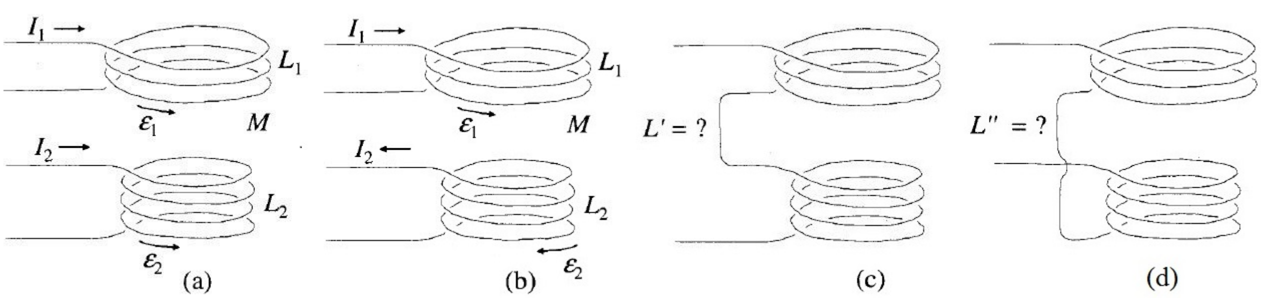
(b) What would happen if we reverse the direction of the positive current and the positive electromotive force in the lower coil (as shown in part (b))?

Now, connect the coils as shown in part (c) to form a single circuit.

(c) What is the inductance L' of this circuit, expressed in terms of L_1, L_2 and M?

(d) What is the inductance L'' of the circuit formed by the coils shown in (d)? Between the circuits shown in (c) and (d), which one has the greater self-inductance?

(e) Given a specific L_1 and L_2, what is the range of possible values of M?



2. A charged sphere

A region in space contains a total positive charge Q that is distributed spherically such that the volume charge density $\rho(r)$ is given by:

$$\rho(r) = \begin{cases} \alpha & : \quad r \leq R/2 \\ 2\alpha\left(1 - \frac{r}{R}\right) & : \quad R/2 \leq r \leq R \\ 0 & : \quad r \geq R, \end{cases} \tag{3}$$

where α is a positive constant. Express your answers below in terms of Q and R.

(a) Determine α.

(b) Find the magnitude of the electric field everywhere.

(c) An electron with charge $q' = -e$ and mass m is placed at $t = 0$ at a position r, with $0 < r \leq R/2$. Describe qualitatively the motion of the electron and find the time, T, when the electron first returns to its starting position.

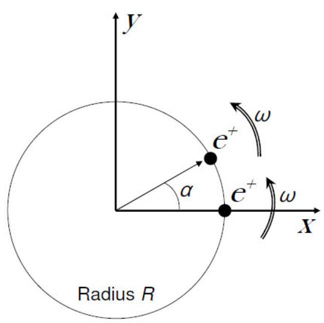
3. Radiation

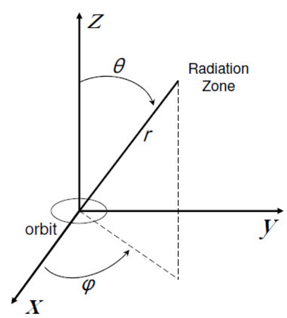
Two point particles (each having the same electric charge $+e$) travel in the xy-plane around the circumference of a circle (having radius R). Both charges travel at the same constant angular velocity ω but maintain a fixed angular separation α throughout the motion. Assume that the motion of the particles is non-relativistic ($\frac{v}{c} = \frac{\omega R}{c} \ll 1$).

(a) Find the electric and magnetic fields (in terms of the unit vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$) produced in the radiation zone at distances r far from the circular orbit (i.e. $r \gg R$).

(b) Find the time-averaged power radiated per unit solid angle in the (θ, ϕ) direction shown in the diagram below.

(c) Explain what would happen if $\alpha = 0$ or $\alpha = \pi$?





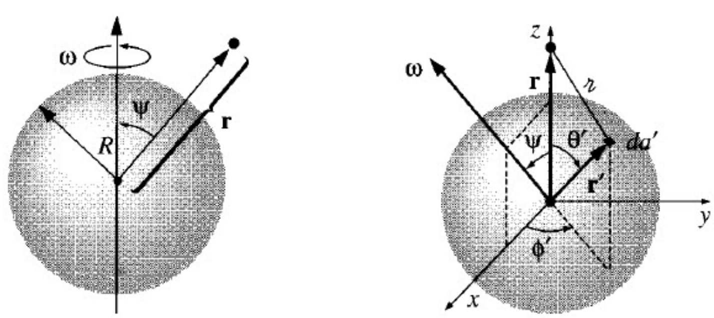
4. A charged spherical shell

A specific charge density $\sigma(\theta) = k\left(3\cos^2\theta - 1\right)$ is glued over the surface of a spherical shell of radius R.

(a) Find the resulting electrostatic potential V inside and outside the sphere.

Now, assume that the sphere has a uniform surface charge distribution, σ, and spins with an angular velocity ω.

(b) Determine the vector potential $\vec{A}(\vec{r})$ inside and outside the sphere. [*Hint:* It will be easier to orient your sphere so that $\vec{r}$ lies on the $\hat{z}$-axis and ω is tilted by angle Ψ in the xz-plane (see Figure below)].



(c) Demonstrate that the magnetic field inside the shell is uniform.

5. Polarization

An optically active medium can rotate the plane of polarization of light. Assume a plane wave propagating in this medium in the z-direction (i.e. the 3-direction) with frequency ω, and the susceptibility tensor of such a medium can be expressed as:

$$\overleftrightarrow{\chi} = \begin{pmatrix} \chi_{11} & i\chi_{12} & 0 \\ -i\chi_{12} & \chi_{11} & 0 \\ 0 & 0 & \chi_{33} \end{pmatrix} \tag{4}$$

where $\overleftrightarrow{\chi}$ is related to the polarizability tensor by $\vec{P} = \varepsilon_0 \overleftrightarrow{\chi} \cdot \vec{E}$. In $\overleftrightarrow{\chi}$, χ_{11}, χ_{12}, and χ_{33} are all real.

(a) Use Maxwell's equations to derive the wave equation for the electric field $\vec{E}$ in the medium.

(b) In the optically active medium, show that the propagating wave is transverse.

(c) Show that the medium admits waves with two distinct k vectors of magnitude k_R and k_L. Find k_R, and k_L in terms of ω and χ_{ij}.

(d) Show that the two k-vectors, k_R and k_L, correspond to the propagation of right and left circularly polarized waves.

(e) Find an expression for the rotary power $\equiv n_R - n_L$ in terms of χ_{ij}.

Equations and Constants

1. Hydrogenic Bohr Radius: $a_0 = 5.3 \times 10^{-11}$ m
2. Mass of Electron: $m_e = 9.1 \times 10^{-31}$ kg
3. Charge of electron: $q_e = 1.6 \times 10^{-19}$ C
4. $\hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$
5. $\hat{y} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}$
6. $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$
7. Speed of light in a medium: $v = \frac{\omega}{k}$
8. $V(r) = \sum\limits_l \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$, where $P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)$ and
9. $\int\limits_0^\pi P_l(\cos\theta)P_m(\cos\theta)\sin\theta\,d\theta = \begin{cases} 0 & if \quad m \neq l \\ \frac{2}{2l+1} & if \quad m = l \end{cases}$
10. $P_0(x) = 1$; $P_1(x) = x$; $P_2(x) = \frac{3x^2-1}{2}$; $P_3(x) = \frac{5x^3-3x}{2}$
11. Electric Dipole Radiation: $\frac{\mu_0}{4\pi r}\left[\hat{r} \times \left(\hat{r} \times \ddot{\vec{p}}\right)\right]$
12. Magnetic field: $\vec{B} = \hat{r} \times \vec{E}$
13. $\frac{dP}{d\omega} = \frac{1}{\mu_0 c}|E|^2 r^2$
14. $\vec{\nabla} \cdot \vec{v} = \partial_x v_x + \partial_y v_y + \partial_z v_z$ (cartesian);
15. $\vec{\nabla} \cdot \vec{v} = \frac{1}{r^2}\partial_r\left(r^2 v_r\right) + \frac{1}{r\,\sin\theta}\partial_\theta\left(r\,\sin\theta\,v_\theta\right) + \frac{1}{r\,\sin\theta}\partial_\phi v_\phi$ (spherical)
16. $d\vec{a} = s\,d\phi\,dz\,\hat{r} + ds\,dz\,\hat{\phi} + s\,ds\,d\phi\,\hat{z}$ (cylindrical)
17. $d\vec{a} = r^2\sin\theta\,d\theta\,d\phi\,\hat{r} + r\,dr\,d\phi\,\hat{\theta} + r\sin\theta\,dr\,d\theta\,\hat{\phi}$ (spherical)
18. $\int \sin^3(ax)\,dx = -\frac{1}{a}\cos(ax) + \frac{1}{3a}\cos^3(ax)$