Classical Mechanics

1. Consider a cylindrically symmetric object with a total mass M and a finite radius R from the axis of symmetry as in the FIG. 1.

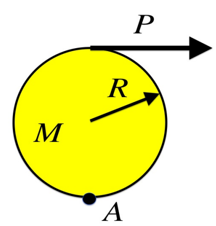


FIG. 1. Figure for (a), (b) and (c).

   (a) Show that the moment of inertia about the symmetry axis I is less than or equal to MR^2, that is, $I \leq MR^2$.

   (b) Suppose the object is initially at rest without motion in an inertial frame of reference. A sudden impulse P is applied at a point on the surface along the tangential direction as shown in the figure. Find the velocity of the center of mass (V) and the angular frequency of rotation (ω) afterwards. Write your answer in terms of P, I, M and R.

   (c) Find the instantaneous velocity of the point A which is at the opposite side of the point of impulse around the symmetry axis as in FIG. 1, and show that its direction is opposite to that of the original impulse P.

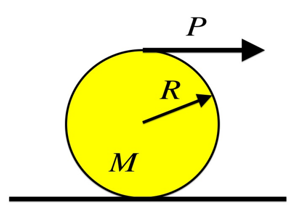


FIG. 2. Figure for (d) and (e).

   (d) Suppose the object is initially at rest, but this time it is on a horizontal surface with friction so that there is no slipping at the point of contact with the surface as in the FIG. 2. A sudden impulse P is applied as before along the horizontal direction at the top point. Find the final velocity V_F.

   (e) Show or argue that $V_F \geq V$, where V is the velocity in the problem (b) and V_F is the velocity in the problem (d). Where does the additional momentum in the problem (d) compared to problem (b) come from?

2. An elementary relativistic particle of rest mass m_i moving with relativistic velocity v_i in the lab frame collides with another identical particle which is at rest in the lab frame, and they become a new single particle of rest mass M as in FIG. 3. Shortly afterwards, it decays into two identical particles of rest mass m_f which fly apart with the same speed v_f with an angle θ from the original direction as shown in FIG. 4. If you prefer, you can use the unit system where the speed of light is unity, $c = 1$.

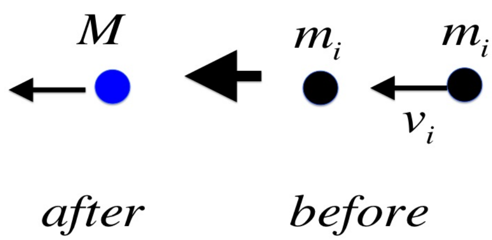


FIG. 3. Figure for (a) and (b).

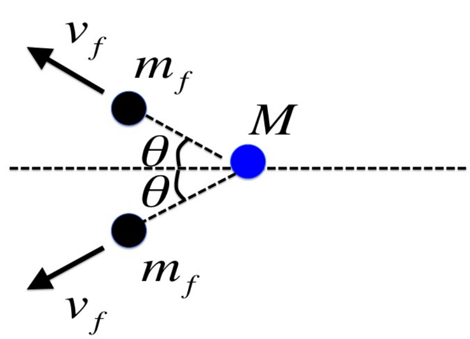


FIG. 4. Figure for (c).

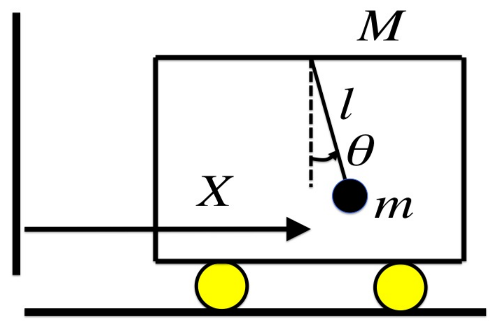
(a) Find the rest mass M of the intermediate particle before its decay, in terms of m_i and v_i.

(b) Find the velocity of the intermediate particle in the lab frame before its decay, in terms of m_i and v_i.

(c) Find the final velocity v_f and the angle θ in the lab frame, in terms of m_i, v_i, and m_f.

(d) Show or argue that in the center of mass frame, the final two particles in this case fly apart in a direction perpendicular to the original incident direction.

3. A train car of total mass M is on the frictionless rail. Inside the car, there is a pendulum of length l with a bob of mass m as shown in the figure. Let the constant acceleration of gravity be g. Assume that the angle of pendulum motion θ is sufficiently small, $\theta \ll 1$, so keep only quadratic order in θ in the Lagrangian. Let the horizontal displacement of the train car be denoted as X.



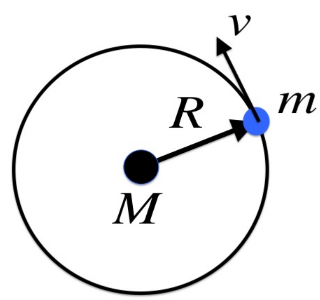
(a) Write down the Lagrangian of the system with generalized coordinates X and θ.

(b) Write down the equations of motion.

(c) Find the two normal modes of the system and their respective normal mode frequencies.

(d) Draw or explain qualitatively the motion corresponding to each normal mode.

4. A small meteoroid of mass m is in the circular orbit of radius R around a very heavy stellar object of mass M. Let the Newton's gravitational constant be G.

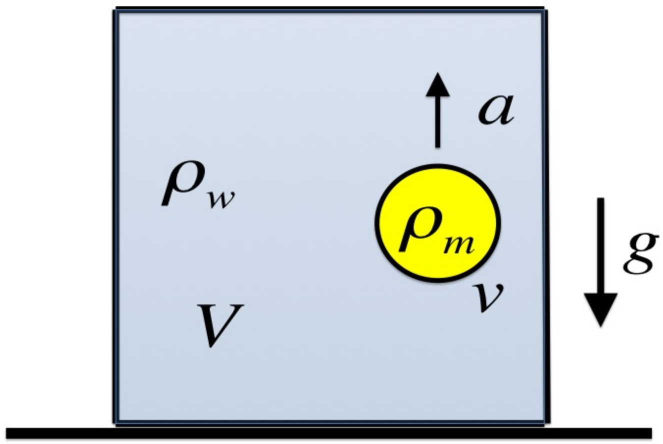


(a) Find the speed v of the meteoroid, its angular momentum L, and its total mechanical energy E.

(b) Suppose the stellar object suddenly explodes in a spherical symmetric way without gaining any momentum, and loses part of its mass to a region far away from the system, and becomes a neutron star of mass $M_f < M$. Assume that the explosion happens very fast compared to the meteoroid motion, and the meteoroid motion is not disrupted during the process, and the meteoroid moves in the gravitational field of the newly born neutron star. Find the minimum value of M_f that the meteoroid orbit is still bound to the neutron star.

(c) Supposing that $M_f = \frac{3}{4}M$, find the farthest distance from the neutron star R_m and the velocity at that point v_m.

5. Consider a big tank of water of total volume V with water density ρ_w, inside of which there is a small object of volume v with mass density $\rho_m < \rho_w$ as in the figure. The whole system is sitting on a stable ground. Let the constant acceleration of gravity be g, and assume that the mass of the tank itself is negligibly small.



(a) Since $\rho_m < \rho_w$, the object is accelerated upward by buoyancy. Find the value of the buoyancy force on the object and the net acceleration a. Ignore any effect of viscosity.

(b) As the object moves upward, it displaces a portion of water downward, causing the center of mass of water as well as the center of mass of the whole system to move. Find the value of the acceleration of the center of mass of the whole system, and show that it is pointing downward.

(c) What is the total gravity force acting on the whole system?

(d) Find the normal force that the ground surface is acting on the tank.

Solutions

1. (a) The moment of inertia is given by $I = \int dm\, r^2$, and since $r \leq R$, we have $I \leq R^2 \int dm = MR^2$.

   (b) The center of mass velocity V is determined by considering the total momentum MV being equal to the impulse P: $V = P/M$. To find the angular velocity ω, we consider the total angular momentum $I\omega$ being equal to the rotational impulse PR: $\omega = PR/I$.

   (c) The velocity at the opposite point along the direction to the original impulse is given by $V - R\omega$, where $-R\omega$ is the velocity of that point due to the angular rotation with angular velocity ω. Using the results in (a) and (b), we have $P/M - PR^2/I = \frac{P}{MI}(I - MR^2) < 0$, which means that the velocity is in opposite direction to the original impulse.

   (d) In this case, the instantaneous point of contact with the surface should have zero velocity due to friction, which gives a constraint $V_F = R\omega_F$ where V_F and ω_F are the final velocity and angular velocity. We should consider the counter impulse from the friction, which we denote P_f. The total momentum should be equal to the total impulse $P + P_f$, which gives $MV_F = P + P_f$, while the total angular momentum should be equal to the total angular impulse $R(P - P_f)$, which gives $I\omega_F = R(P - P_f)$. Solving these, we get

$$V_F = \frac{2P}{M + I/R^2}\,, \quad P_f = P\,\frac{M - I/R^2}{M + I/R^2} > 0\,. \tag{0.1}$$

   (e) Since $M > I/R^2$, we have $V_F = \frac{2P}{M+I/R^2} > \frac{2P}{M+M} = P/M = V$. The additional momentum comes from the frictional impulse $P_f > 0$ which is along the same direction of the original implies P. Intuitively, from (c), since the velocity at the point of contact without surface is opposite to the original direction of impulse, the friction force should point to the same direction to the original impulse.

2. (a) The initial total energy is $E_i = m_i(\gamma_i + 1)c^2$, where $\gamma_i = 1/\sqrt{1 - v_i^2/c^2}$, and the initial total momentum is $p_i = m_i\gamma_i v_i$. Using the fact that $(E/c^2)^2 - (p/c)^2$ is Lorentz invariant, and should be equal to M^2 in the center of mass frame, we have

$$M^2 = (E_i/c^2)^2 - (p_i/c)^2 = m_i^2((\gamma_i + 1)^2 - \gamma_i^2 v_i^2/c^2) = 2m_i^2(\gamma_i + 1)\,. \qquad (0.2)$$

(b) Let the velocity of the intermediate particle be v. The momentum of the intermediate particle $M\gamma v$ where $\gamma = 1/\sqrt{1 - v^2/c^2}$ should be equal to the initial momentum $p_i = m_i\gamma_i v_i$. Squaring this equality we have

$$\frac{v^2}{1 - v^2/c^2} = \frac{m_i^2\gamma_i^2 v_i^2}{M^2} = \frac{m_i^2\gamma_i^2 v_i^2}{2m_i^2(\gamma_i + 1)} = \frac{(\gamma_i - 1)}{2}c^2\,. \qquad (0.3)$$

Solving this for v we get

$$v^2/c^2 = \frac{(\gamma_i - 1)}{(\gamma_i + 1)}\,. \qquad (0.4)$$

(c) The momentum conservation gives $2m_f\gamma_f v_f \cos\theta = m_i\gamma_i v_i$, and the energy conservation gives $2m_f\gamma_f = m_i(\gamma_i + 1)$, where $\gamma_f = 1/\sqrt{1 - v_f^2/c^2}$. From the second equation, we easily solve for v_f to be

$$v_f^2/c^2 = 1 - \frac{4m_f^2}{m_i^2(\gamma_i + 1)^2}\,. \qquad (0.5)$$

Using the relation

$$\gamma_f^2 v_f^2 = (\gamma_f^2 - 1)c^2 = \left(\frac{m_i^2(\gamma_i + 1)^2}{4m_f^2} - 1\right)c^2\,, \qquad (0.6)$$

and squaring the first equation, we get

$$\left(m_i^2(\gamma_i + 1)^2 - 4m_f^2\right)\cos^2\theta = m_i^2(\gamma_i^2 - 1)\,, \qquad (0.7)$$

so we have

$$\cos^2\theta = \frac{m_i^2(\gamma_i^2 - 1)}{m_i^2(\gamma_i + 1)^2 - 4m_f^2}\,. \qquad (0.8)$$

(d) If we consider the center of mass frame, the final two particles should fly apart collinear to each other. In the lab frame they make the common angle θ to the incident direction, which means that the angle the final particles make to the incident direction in the center of mass frame should also be common. This is only possible when they fly apart in perpendicular direction to the incident direction in the center of mass frame.

3. (a) The kinetic energy for the train car is $\frac{1}{2}M\dot{X}^2$. To find the kinetic energy of the pendulum, the x-position of the bob of mass is

$$x = X + l\sin\theta \approx X + l\theta\,, \tag{0.9}$$

while the y-position is $y = -l\cos\theta \approx -l + \frac{1}{2}l\theta^2$. Therefore the kinetic energy of the bob of mass in quadratic in θ is $\frac{1}{2}m(\dot{X} + l\dot{\theta})^2$. On the other hand, the potential energy is from gravity which is $V = mgy \approx -mgl + \frac{1}{2}mgl\theta^2$. We can neglect the constant $-mgl$ in the potential energy. Therefore, the Lagrangian of the system is

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X}+l\dot{\theta})^2 - \frac{1}{2}mgl\theta^2 = \frac{1}{2}(M+m)\dot{X}^2 + ml\dot{X}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}mgl\theta^2\,. \tag{0.10}$$

(b) The equation of motion from X is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) - \frac{\partial L}{\partial X} = (M+m)\ddot{X} + ml\ddot{\theta} = 0\,, \tag{0.11}$$

while the equation of motion from θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = ml\ddot{X} + ml^2\ddot{\theta} + mgl\theta = 0\,. \tag{0.12}$$

(c) To find the normal modes and their frequencies, we write the above equations of motion in terms of matrix equation, while replacing $\frac{d}{dt} \to -i\omega$. We get

$$\begin{pmatrix} -(M+m)\omega^2 & -ml\omega^2 \\ -ml\omega^2 & -ml^2\omega^2 + mgl \end{pmatrix} \begin{pmatrix} X \\ \theta \end{pmatrix} = 0\,. \tag{0.13}$$

Requiring that the above matrix has zero determinant to have a non-zero solution of (X, θ), we get the equation for ω^2;

$$\omega^2\left(Ml\omega^2 - (M+m)g\right) = 0\,, \tag{0.14}$$

that gives two normal modes frequencies $\omega^2 = 0$ or $\omega^2 = \frac{(M+m)}{M}g/l$. For $\omega^2 = 0$, the normal mode is easily found to be

$$\begin{pmatrix} X \\ \theta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \tag{0.15}$$

and for $\omega^2 = \frac{(M+m)}{M}g/l$, the normal mode is

$$\begin{pmatrix} X \\ \theta \end{pmatrix} = \begin{pmatrix} ml \\ -(M+m) \end{pmatrix}\,. \tag{0.16}$$

(d) The first normal mode is a constant motion of the whole system along x-direction with the pendulum at zero angle. This clearly is a solution of equation of motion due to Galilean invariance. The second normal mode is a simple harmonic motion where the train car and the pendulum oscillate in complete out-of phase to each other, so that the center of mass of the total system does not move at all.

4. (a) Equating the gravity force GMm/R^2 with the centripetal force mv^2/R, we have $v = \sqrt{GM/R}$. The angular momentum is $L = mRv = m\sqrt{GMR}$ and the total mechanical energy is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{R} = -\frac{1}{2}\frac{GMm}{R} < 0\,. \tag{0.17}$$

(b) Since the motion of the meteoroid is not affected by the fast explosion, we can use the same tangential velocity $v = \sqrt{GM/R}$ right after the collision. The kinetic energy is therefore the same: $KE = \frac{1}{2}\frac{GMm}{R}$, while the gravitational potential energy is now given by M_f, $VE = -\frac{GM_f m}{R}$. In order for the motion to be bounded, the total mechanical energy has to be negative,

$$KE + VE = \frac{1}{2}\frac{GMm}{R} - \frac{GM_f m}{R} < 0\,, \tag{0.18}$$

which gives the condition $M_f > \frac{1}{2}M$. The minimal mass is therefore $\frac{1}{2}M$.

(c) Right after the explosion, the angular momentum is still given by $L = m\sqrt{GMR}$, and the total mechanical energy is

$$E = \frac{1}{2}\frac{GMm}{R} - \frac{GM_f m}{R} = \frac{1}{2}\frac{GMm}{R} - \frac{3}{4}\frac{GMm}{R} = -\frac{1}{4}\frac{GMm}{R}\,. \tag{0.19}$$

These two have to be conserved. At the farthest distance, the angular momentum and energy conservation give the two equations

$$m\sqrt{GMR} = mR_m v_m\,, \quad -\frac{1}{4}\frac{GMm}{R} = \frac{1}{2}mv_m^2 - \frac{GM_f m}{R_m} = \frac{1}{2}mv_m^2 - \frac{3}{4}\frac{GMm}{R_m}\,. \tag{0.20}$$

These two equations involve two unknowns R_m, v_m, and solving these, we get

$$R_m = 2R\,, \quad v_m = \frac{1}{2}\sqrt{\frac{GM}{R}}\,. \tag{0.21}$$

5. (a) The buoyancy force is upward and is given by $F_b = \rho_w v g$. Since there is gravity force $F_g = -\rho_m v g$ acting downward, the net force upward is $F = (\rho_w - \rho_m) v g$ which should be equal to $\rho_m v a_m$ where a_m is the net acceleration upward. This gives $a_m = (\rho_w/\rho_m - 1)g$.

(b) Let the height of the object from the ground be x_m, and the height of the center of mass of the water be x_w. Then, the height of the center of mass of the whole system is

$$x_{CM} = \frac{x_m \rho_m v + x_w \rho_w V}{\rho_m v + \rho_w V}\,, \tag{0.22}$$

so the acceleration is

$$a_{CM} = \frac{a_m \rho_m v + a_w \rho_w V}{\rho_m v + \rho_w V}\,. \tag{0.23}$$

Since we know a_m in (a), what we need to find is the acceleration of the center of mass of the water a_w. To find this, notice that $(V - v)$ volume of water stays the same while v volume of water is accelerating down by the amount $-a_m$. Therefore, we have

$$a_w = \frac{(V - v)\cdot 0 - v a_m}{V} = -\frac{v}{V} a_m\,. \tag{0.24}$$

This gives the net downward acceleration of the whole system as

$$a_{CM} = a_m \frac{(\rho_m - \rho_w) v}{\rho_m v + \rho_w V} = -\frac{(\rho_w - \rho_m)^2 v g}{\rho_m(\rho_m v + \rho_w V)}\,. \tag{0.25}$$

(c) The total gravity force acting on the whole system is simply $F_g = -(\rho_m v + \rho_w V)g$ which is downward.

(d) With the normal force N upward, the total force acting on the system is $F_g + N = -(\rho_m v + \rho_w V)g + N$. This should be equal to the total mass of the system times the acceleration of the center of mass a_{CM}: $(\rho_m v + \rho_w V)a_{CM} = -\frac{(\rho_w - \rho_m)^2 v g}{\rho_m}$. Therefore, the normal force is

$$N = (\rho_m v + \rho_w V)g - \frac{(\rho_w - \rho_m)^2 v g}{\rho_m} = \rho_w V g + (2\rho_w \rho_m - \rho_w^2)\frac{v g}{\rho_m}\,. \tag{0.26}$$