1. Gas in magnetic field

Consider monoatomic ideal (Boltzmann) gas in which the atoms carry magnetic moment μ. The gas is placed in magnetic field B pointing up. Use units in which $k_B = 1$. At a given temperature T,

(a) Find the average fraction of atoms, f_+ with magnetic moment pointing up and the fraction of atoms, f_-, pointing down.

(b) Find the average energy per atom, U_B, due to the interaction with the magnetic field.

(c) Find the heat capacity at constant volume and magnetic field, C_V, per atom of the gas. (Use equipartition theorem for the translational motion of the atoms).

(d) Find the entropy S per atom of the gas (up to an additive constant). Use $\int x dx / \cosh^2 x = x \tanh x - \ln \cosh x$

(e) Find what happens to the temperature of the gas in a thermally isolated container if the magnetic field is reduced adiabatically.

Solutions:

(a)

$$f_+ = \frac{e^x}{e^x + e^{-x}}, \quad f_- = \frac{e^{-x}}{e^x + e^{-x}}, \quad \text{where} \quad x = \mu B/T$$

(b)

$$U_B = -\mu B(f_+ - f_-) = -\mu B \tanh\left(\frac{\mu B}{T}\right)$$

(c)

$$C_V = \frac{3}{2} + \frac{\partial U_B}{\partial T} = \frac{3}{2} + \frac{x^2}{\cosh^2 x}$$

(d) Noting that $dT/T = -dx/x$ and using the given integral we find

$$S = \int C_V \frac{dT}{T} = \frac{3}{2} \ln T + \ln \cosh x - x \tanh x + \text{const}$$

(e) The entropy decreases when the magnetic field is increased, i.e., $(\partial S/\partial B)_T < 0$. This can be verified explicitly or argued on the grounds that the spin ordering reduces entropy. Therefore,

$$\left(\frac{\partial T}{\partial B}\right)_S = -\left(\frac{\partial S}{\partial B}\right)_T \left(\frac{\partial T}{\partial S}\right)_B = -\left(\frac{\partial S}{\partial B}\right)_T \frac{1}{C_V} > 0,$$

i.e., T will decrease when B is reduced.

2. Maxwell relations for a spring

A spring is stretched by constant force $\mathcal{F}$ to length L at at given temperature T . The spring constant $k = (d\mathcal{F}/dL)_T$ is given.

(a) Write down the equation expressing the first law of thermodynamics (energy conservation) for the string in infinitesimal form, i.e., express dU in terms of dS and dL.

(b) Express thermodynamic potential Φ for which $\mathcal{F}$ and T are independent variables in terms of U, S, $\mathcal{F}$ and L.

(c) Express $d\Phi$ in terms of $d\mathcal{F}$ and dT.

(d) When the temperature is raised by a small amount ΔT, the spring expands by ΔL. How much heat ΔQ will the spring exchange with the environment when the force is slowly decreased to a value necessary to return the spring to the original length L at constant temperature? Will the heat be emitted or absorbed?

(e) A stretched rubber band behaves somewhat differently: it contracts when the temperature is increased. Will the heat be emitted or absorbed when the force stretching the band is reduced?

Solutions:

(a) $dU = TdS + \mathcal{F}dL$

(b) $\Phi = U - ST - \mathcal{F}L$.

(c) $d\Phi = -SdT - Ld\mathcal{F}$.

(d)

$$\Delta Q = T\Delta S = T\left(\frac{\partial S}{\partial \mathcal{F}}\right)_T \Delta\mathcal{F} = T\underbrace{\left(\frac{\partial L}{\partial T}\right)_{\mathcal{F}}}_{>0}\underbrace{\Delta\mathcal{F}}_{<0} = T\frac{\Delta L}{\Delta T}(-k\Delta L) = -kT\frac{(\Delta L)^2}{\Delta T} < 0.$$

The heat is emitted into the environment.

(e) Since $(dL/dT)_{\mathcal{F}} < 0$, from equation above we find the heat is absorbed.

3. Two particles

Consider a system of two particles each of which can occupy 3 different levels with energies 0, ε and 2ε. Write the partition function for the system if the particles obey

(a) Write the partition function for the system if the particles are distinguishable.

(b) Write the partition function for the system if the particles are indistinguishable and obey Fermi-Dirac statistics.

(c) Find the entropy of the system in part (b) in the limit $T \to \infty$.

(d) Find the energy of the system in part (b) in the limit $T \to \infty$.

(e) Write the partition function for the system if the particles are indistinguishable and obey Bose-Einstein statistics.

Solutions:

(a) $Z = (1 + y + y^2)^2 = 1 + 2y + 3y^2 + 2y^3 + y^4$, where $y = e^{-\varepsilon/T}$.

(b) There are 3 states: 01, 02, 12. $Z = y + y^2 + y^3 = (1 + y + y^2)y$.

(c) All 3 states are equially probable. $S = \ln 3$.

(d) All 3 states are equially probable and their energies are ε, 2ε and 3ε. The average energy is $U = 2\varepsilon$.

(e) There are 6 states: 00, 01, 02, 11, 12, 22. $Z = 1 + y + 2y^2 + y^3 + y^4 = (1 + y + y^2)(1 + y^2)$.

4. Gas in a potential

A gas consisting of N identical classically moving but indistingushable particles is confined inside a cylinder of length $L = a - b$ and crossection area $A = \pi R^2$ defined, in cartezian coordinates, by

$$b < z < a, \qquad x^2 + y^2 < R^2.$$

Particles do not interact with each other and the motion of each is governed by the Hamiltonian

$$H = \frac{\boldsymbol{p}^2}{2m} + Kz$$

where K is a constant. Use units in which $k_B = 1$. The gas is in equilibrium.

(a) Calculate the partition function of the gas.

(b) Determine the pressure p_a of the gas on the wall at $z = a$.

(c) Determine the pressure p_b of the gas on the wall at $z = b$.

(d) Compare the pressures at p_a and p_b in the limit $KL \gg T$. Explain.

(e) What is the root mean square, $\overline{\boldsymbol{v}^2}(z)$, of a particle velocity at a given coordinate z.

Solutions:

(a)

$$Z = \frac{1}{N!}\left[A\int_b^a dz \int d^3\left(\frac{\boldsymbol{p}}{2\pi}\right)\exp\left\{-\frac{\boldsymbol{p}^2}{2mT} - \frac{Kz}{T}\right\}\right]^N = \frac{1}{N!}\left[A\left(\frac{mT}{2\pi}\right)^{3/2}\frac{T}{K}\left(e^{-Kb/T} - e^{-Ka/T}\right)\right]^N$$

(b)

$$p_a = -\frac{\partial F}{A\partial a} = \frac{T\partial \ln Z}{A\partial a} = \frac{TN}{A}\frac{\partial}{\partial a}\ln\left(e^{-Kb/T} - e^{-Ka/T}\right) = \frac{KN}{A}\frac{1}{e^{KL/T} - 1}$$

(c)

$$p_b = +\frac{\partial F}{A\partial b} = -\frac{TN}{A}\frac{\partial}{\partial b}\ln\left(e^{-Kb/T} - e^{-Ka/T}\right) = \frac{KN}{A}\frac{1}{1 - e^{-KL/T}}$$

(d) In the limit $KL \gg T$,

$$p_a \approx \frac{KN}{A}e^{-KL/T} \ll p_b \approx \frac{KN}{A}$$

The density of the particles is $e^{-KL/T}$ smaller at $z = a$ vs $z = b$, because of the potential difference and thus the pressure is smaller.

(e) By equipartion theorem, $\dfrac{\overline{m\boldsymbol{v}^2}}{2} = \dfrac{3T}{2}$ and thus $\overline{\boldsymbol{v}^2} = \dfrac{3T}{m}$ independent of z.

5. Star

Consider electrons in the interior of a cold star as an ideal non-relativistic Fermi gas at zero temperature.

(a) Determine the Fermi momentum of the electrons, p_F, for a given density n of electrons per unit volume.

(b) Find the chemical potential μ of the electrons in terms of n.

(c) For sufficiently high density the Fermi energy of the electrons becomes sufficient to overcome the threshold of a reaction in which an electron is captured by a nucleus:

$$A_Z + e^- \longrightarrow A_{Z-1} + \nu$$

where the neutrino escapes from the star. Given the difference in binding energies $\varepsilon_{A,Z} - \varepsilon_{A,Z-1} = \Delta$, calculate the minimum electron density, n_Δ, required for this reaction to occur. Neglect neutrino mass.

(d) Find the pressure of the electron gas at this threshold density. Express the result in terms of Δ.

Solutions:

(a)

$$n = 2\frac{4\pi p_F^3/3}{(2\pi\hbar)^3} = \frac{p_F^3}{3\pi^2\hbar^3} \qquad \text{thus} \qquad p_F = \hbar(3\pi^2 n)^{1/3}$$

(b) Energy needed to add one more electron is

$$\mu = \frac{p_F^2}{2m_e} = \frac{\hbar^2}{2m_e}(3\pi^2 n)^{2/3}$$

(c) The condition at threshold is $\mu = \Delta$, thus

$$n_\Delta = \frac{(2m_e\Delta)^{3/2}}{3\pi^2\hbar^3}$$

(d) Pressure can be determined by integrating $dp = n d\mu$. At threshold $\mu = \Delta$, thus

$$p_\Delta = \int_0^\Delta \frac{(2m_e\mu)^{3/2}}{3\pi^2\hbar^3} d\mu = \frac{2(2m_e)^{3/2}}{15\pi^2\hbar^3}\Delta^{5/2}$$