Quantum Mechanics Problems

1. *Parts* **(a)** *and* **(b)** *of this problem are independent of each other.*

   (a) Let A, B, and C be Hermitian operators on a Hilbert space. Assume that operator A has a non-degenerate spectrum and that AB and AC are Hermitian. **(i)** Prove that $[B, C] = 0$. **(ii)** If $|\psi_1\rangle$ and $|\psi_2\rangle$ are eigenstates of B with distinct eigenvalues, show that $\langle\psi_1|C|\psi_2\rangle = 0$.

   (b) Consider a quantum system whose Hamiltonian operator is given by $H = A^2 + B^2$, where A and B are two Hermitian operators. If $[A, B] = -i\alpha\hat{I}$, where α is a positive number and $\hat{I}$ is the identity operator, show that the ground state energy E_0 of the system satisfies $E_0 \geq \alpha$. If this inequality is saturated (i.e. $E_0 = \alpha$), show that the ground state ket $|\psi_0\rangle$ satisfies $A|\psi_0\rangle = \lambda B|\psi_0\rangle$ for some λ to be determined. (Suggestion: Consider the operator $C = A - iB$)

2. *Parts* **(a)** *and* **(b)** *of this problem are independent of each other.*

   (a) Consider a particle of mass m moving in the one-dimensional potential $V(x) = \Lambda\delta(x)$ for $|x| < a$ and $V(x) = \infty$ elsewhere. **(i)** Find the value of Λ for which the ground state energy of the system vanishes. **(ii)** Find the energy of the first excited state of the system. **(iii)** Sketch the wavefunction for the second excited state. Explain your reasoning clearly for all parts.

   (b) For a particle of mass m moving in the one-dimensional harmonic oscillator potential $\frac{1}{2}m\omega^2x^2$, the Hamiltonian eigenstates can be written in terms of $\eta = x\sqrt{\frac{m\omega}{\hbar}}$ as $\psi_n(x) = N_n H_n(\eta)e^{-\eta^2/2}$ where N_n is a normalization constant and $H_n(\eta)$ is the Hermite polynomial of degree n. The first few Hermite polynomials are provided on the next page. Consider a particle of mass m moving in a harmonic oscillator potential inside an infinite-potential box of width $2a$. That is, $V(x) = \frac{1}{2}m\omega^2x^2$ for $|x| < a$ and $V(x) = \infty$ elsewhere.
   (i) Find the ground state energy of the system if $a = \sqrt{\hbar/2m\omega}$. **(ii)** Find the first excited state energy of the system if $a = \sqrt{3\hbar/2m\omega}$. Explain your reasoning clearly for both parts.

3. *Parts* **(a)** *and* **(b)** *of this problem are independent of each other.*

   (a) Consider a quantum system described by the following Hamiltonian on a three-dimensional Hilbert space with an orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$, where $b \ll 1$ is a small parameter.

   $$H = -|1\rangle\langle 1| - |2\rangle\langle 2| + b\Big(|1\rangle\langle 2| + |2\rangle\langle 1| + |1\rangle\langle 3| + |3\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|\Big)$$

   Using perturbation theory, compute the energy levels of the system up to second order in b.

   (b) Consider another quantum system described by the following Hamiltonian on a three-dimensional Hilbert space with an orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$, where $b \ll 1$ is a small parameter.

   $$H = -|1\rangle\langle 1| + |3\rangle\langle 3| + b\Big(|1\rangle\langle 2| + |2\rangle\langle 1|\Big) + 2b\Big(|1\rangle\langle 3| + |3\rangle\langle 1|\Big)$$

   Compute the ground state wavefunction of the system up to first order in b, and use this wavefunction as a trial wavefunction to get a variational estimate for the ground state energy of the system correct to order b^2. Show that this estimate coincides with the ground state energy of the system computed perturbatively up to second order in b.

4. Let $\phi_{n\ell m}(\mathbf{r})$ denote the normalized Hamiltonian eigenstates for an electron moving in the Coulomb potential of a proton, where n, ℓ, and m denote the principal, angular momentum, and magnetic quantum numbers, respectively. Assume that the electron is in the following normalized superposition of such eigenstates:

$$\psi(\mathbf{r}) = \frac{\sqrt{3}}{4}\phi_{321}(\mathbf{r}) - \frac{i\sqrt{3}}{2\sqrt{2}}\phi_{42-2}(\mathbf{r}) + \frac{1}{2}\phi_{21-1}(\mathbf{r}) + \frac{\sqrt{3}}{4\sqrt{2}}\phi_{520}(\mathbf{r}) - \frac{i\sqrt{3}}{4\sqrt{2}}\phi_{100}(\mathbf{r})$$

(a) If measurements of the energy E, angular momentum squared L^2, and parity Π are performed on this system, write down the possible values that can be obtained and calculate the corresponding probabilities for each outcome.

(b) Calculate $(\Delta L_z)_\psi$, the uncertainty of L_z in the state $\psi(\mathbf{r})$.

(c) Suppose L^2 is measured at $t = 0$ and the largest possible value is obtained. If L_z is measured at a later time t, what are probabilities of the various outcomes as a function of time?

(d) Suppose an energy measurement is performed, and the highest possible value is obtained. Sketch the radial probability distribution $P(r)$ of the electron after this measurement. Here, $P(r)$ is defined such $P(r)dr$ is the probability of finding the electron in a spherical shell between r and $r + dr$. The asymptotic behavior of $P(r)$ near the origin is $P(r \to 0) \sim r^a$. What is the value of a?

(e) Suppose an energy measurement is performed, and the lowest possible value is obtained. Sketch the radial wavefunction and the radial probability distribution after this measurement. At what value of r is the maximum of $P(r)$ achieved?

5. *Parts* **(a)** *and* **(b)** *of this problem are independent of each other.*

(a) Consider two non-interacting electrons of mass m in a one-dimensional infinite potential well with edges placed at $x = 0$ and $x = a$. Write down the energies, the normalized wavefunctions (including both the spatial and spin parts), and the degeneracies of the ground, first excited, and second excited states of the system. In writing the spin part, use the notation *e.g.* $|+\rangle|-\rangle$ to denote the state in which the spin of the first electron is up and that of the second electron is down.

(b) Two spin 1 objects interact with an external vector field $\vec{A}$ in such a way that the Hamiltonian for the system can be written as

$$H = \vec{A}\cdot(\vec{S}_1 - \vec{S}_2).$$

A measurement of the *total* spin angular momentum of the two-particle system performed at $t = 0$ yields a value of 0. Calculate the normalized state of the system $|\psi(t)\rangle$ at a later time t, as well as the first time $|\psi(t)\rangle$ becomes orthogonal to $|\psi(0)\rangle$.

Possibly Useful Information

- The first few Hermite polynomials $H_n(\eta)$ are $H_0(\eta) = 1$, $H_1(\eta) = 2\eta$, $H_2(\eta) = 4\eta^2 - 2$, $H_3(\eta) = 8\eta^3 - 12\eta$, $H_4(\eta) = 16\eta^4 - 48\eta^2 + 12$, $H_5(\eta) = 32\eta^5 - 160\eta^3 + 120\eta$.
- For $s_1 = s_2 = 1$, some of the $|S, M\rangle$ in the coupled representation are given in terms of the uncoupled basis vectors $|m_1\rangle|m_2\rangle$ as (using the notation $+$, 0, $-$ to denote $m_i = +1, 0, -1$, respectively)

$$\begin{aligned}
|2,0\rangle &= \frac{1}{\sqrt{6}}\Big(|+\rangle|-\rangle + 2|0\rangle|0\rangle + |-\rangle|+\rangle\Big) \\
|1,0\rangle &= \frac{1}{\sqrt{2}}\Big(|+\rangle|-\rangle - |-\rangle|+\rangle\Big)
\end{aligned}$$