University of Illinois at Chicago
Department of Physics

Electricity and Magnetism
PhD Qualifying Examination

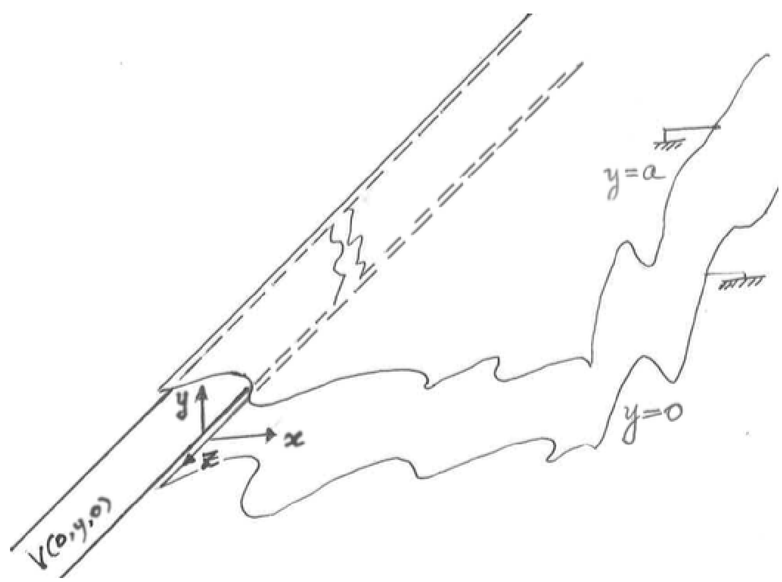
January 8, 2016 (Friday)
9:00 am - 12:00 noon

Full credit can be achieved from completely correct answers to **4 questions**. If the student attempts all 5 questions, all of the answers will be graded, and the **top 4 scores** will be counted toward the exam's total score.

1. Two semi-infinite grounded metal plates are parallel to the xz plane, one at $y = 0$, the other at $y = a$. The left end of their gap, at $x = 0$, is capped off with an infinitely long strip insulated from the two plates and maintained at a specific z-independent potential $V(0, y, z)$. By the method of separation of variables, the potential inside this "slot" $(x > 0)$ is known to be

$$V(x, y, z) = \sum_{n=1}^{\infty} C_n e^{-k_n x/a} \sin(n\pi y/a).$$

- Explicitly find the exponent coefficient k_n in terms of a and n based on the Laplace equation.
- Now if the left end potential is explicitly set up to be $V(0, y, z) = V_0 \sin(7\pi y/a)$, determine all C_n *simply by inspection*. Explicitly find the charge residing in the area between $z = 0$ and $z = b$ on the lower xz plane $(y = 0)$, in terms of V_0, a, b, ϵ_0.
- Explicitly determine the resultant electrostatic force acting on the above mentioned area.



The Laplace equation implies $(\frac{k_n}{a})^2 - (\frac{n\pi}{a})^2 = 0$. So, $\boxed{k_n = n\pi}$. The given potential $V(0, y, z) = V_0 \sin(7\pi y/a)$ simply sets $\boxed{C_7 = V_0, \text{ otherwise } C_n = 0}$.

$$\sigma(x, 0, z) = \epsilon_0 E_y = -\epsilon_0 V_0 (\tfrac{7\pi}{a}) e^{-\frac{7\pi x}{a}} \ .$$

$$Q(z \in [0, b]) = -b\epsilon_0 V_0 \int_0^{\infty} (\tfrac{7\pi}{a}) e^{-\frac{7\pi x}{a}} dx = -b\epsilon_0 V_0 \ .$$

$$F = b \int_0^{\infty} \tfrac{\sigma^2}{2\epsilon_0} dx = \tfrac{1}{2} b\epsilon_0 V_0^2 \int_0^{\infty} (\tfrac{7\pi}{a})^2 e^{-\frac{14\pi x}{a}} dx = \tfrac{7\pi}{4a} b\epsilon_0 V_0^2 \ .$$

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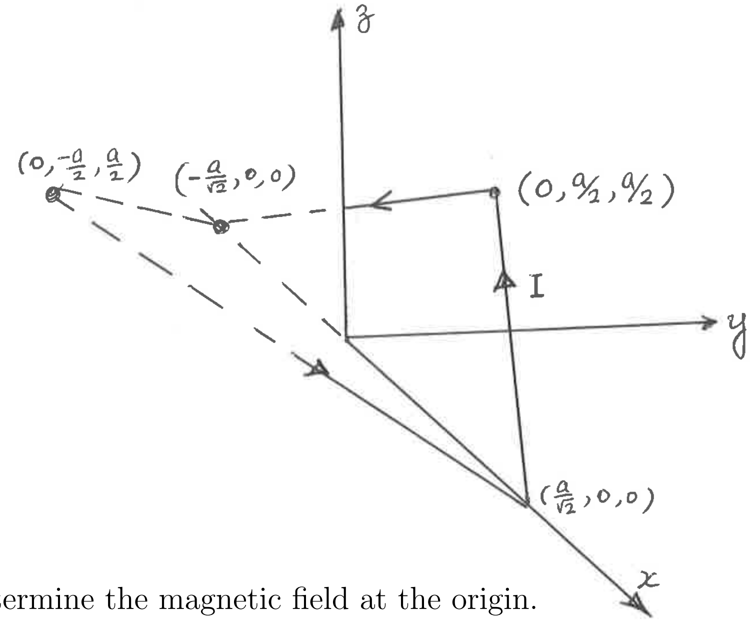
2. A square circuit $a \times a$ is folded along the diagonal line into two perpendicular isosceles right triangles. The fold-line is placed along the x axis. The center of the circuit is placed at the origin O. The current I flows around the first isosceles, starting from $(a/\sqrt{2}, 0, 0)$, reaching $(0, a/2, a/2)$, then $(-a/\sqrt{2}, 0, 0)$. The current moves around the second isosceles, flowing to $(0, -a/2, a/2)$, and returning to the beginning point.



- Determine the magnetic field at the origin.
- Determine the *leading* dipole magnetic field $\boldsymbol{B}$ at a large distance $r \gg a$.
- A secondary circular loop is placed at the spherical coordinates of fixed $r = R \gg a$, $\theta = 60°$ and $\phi \in [0, 2\pi]$. Find the leading dipole contribution of the mutual inductance between the two circuits.
- If the secondary loop carries another current I', determine the leading contribution of the magnetic flux due to I' through the first small circuit.

For an unfolded square circuit, $B_{\text{square}} = \frac{2\sqrt{2}\mu_0 I}{\pi a}$ by Biot Savart law. It is straightforward to show that for our case of the folded square, $\boxed{\boldsymbol{B}_{\text{folded}}(0) = \frac{2\mu_0 I}{\pi a}\hat{\boldsymbol{z}}}$. The magnetic dipole is $\boldsymbol{m} = (a^2 I/\sqrt{2})\hat{\boldsymbol{z}}$. So, for the far field,

$$\boldsymbol{B}(r, \theta) = \frac{\mu_0 \frac{a^2 I}{\sqrt{2}}}{4\pi} \left(\frac{2\cos\theta}{r^3}\hat{\boldsymbol{r}} + \frac{\sin\theta}{r^3}\hat{\boldsymbol{\theta}} \right) .$$

Therefore, the flux through the secondary loop is

$$\Phi_{2\leftarrow 1} = 2\pi R^2 \int B_r(R) d(\cos\theta) = \frac{\mu_0 I a^2}{\sqrt{2}R} \int_{\frac{1}{2}}^{1} \cos\theta (d\cos\theta) = \frac{3\mu_0 I a^2}{8\sqrt{2}R} , \quad M_{21} = \frac{3\mu_0 a^2}{8\sqrt{2}R} .$$

As $M_{21} = M_{12}$ reciprocally, the flux through the first circuit due to I' is

$$\Phi_{1\leftarrow 2} = \frac{3\mu_0 I' a^2}{8\sqrt{2}R} .$$

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3. (i) Determine the electric flux of the Coulomb field from a point charge q located at $(0,0,z)$ through a circular disk of radius R, on the xy plane and centered at $(0,0,0)$ with its normal $\hat{\boldsymbol{n}} = +\hat{\boldsymbol{z}}$. Show the result $\Phi_e \equiv \int_{\text{disk}} \boldsymbol{E} \cdot \hat{\boldsymbol{n}} d^2a$ explictly in terms of q, R, z. Describe any discontinuity.

(ii) The point charge q moves at a uniform velocity v along the z axis. Its position is $z = vt$. In the non-relativistic limit for the slow motion, the corresponding electric field is simply given by the instantaneous Coulomb form.

Determine the Maxwell displacement current I_d through the disk area, in terms of q, R, v, t. Use appropriate mathematical prescription for the singular behavior in the interval when $t \approx 0$.

(iii) Summing up the displacement current I_d and the particle q current $I_q = qv\delta(vt)$, find the Amperian integral $\oint \boldsymbol{B} \cdot d\boldsymbol{\ell}$, in terms of q, R, v, t.

(iv) Explicitly give the magnetic field at $(R, 0, 0)$ based on above reasoning and the symmetry, for all t.

Direct calculation of the Coulombic flux gives

$$\Phi_e = \frac{q}{4\pi\epsilon_0}\int_0^{R^2} \frac{-z\pi d(r^2)}{(z^2+r^2)^{\frac{3}{2}}} = \frac{qz}{2\epsilon_0}\left[\frac{1}{\sqrt{z^2+R^2}} - \frac{1}{|z|}\right] .$$

Similar result can be obtained from the solid angle evaluation.

$$\Omega = -\text{sign}(z)2\pi[1 - |z|/\sqrt{R^2+z^2}] ,$$

$$\Phi_e(t) = -\tfrac{q}{2\epsilon_0}\text{sign}(vt) + \tfrac{q}{2\epsilon_0}vt/\sqrt{R^2+(vt)^2} .$$

$$\epsilon_0 \tfrac{d}{dt}\Phi_e(t) = -qv\delta(vt) + \tfrac{1}{2}qvR^2/[R^2+(vt)^2]^{\frac{3}{2}} .$$

$$\oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_0(I_q + I_d) = \tfrac{1}{2}qvR^2/[R^2+(vt)^2]^{\frac{3}{2}}$$

$$\boldsymbol{B}(R,0,0) = \tfrac{\mu_0 qvR}{4\pi}\hat{\boldsymbol{y}}/[R^2+(vt)^2]^{\frac{3}{2}} .$$

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4. An accelerated charged particle q radiates the retarded electric field, approximately given by

$$\boldsymbol{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0 c^2}\left(\frac{\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \boldsymbol{a})}{r}\right) ,$$

where $\boldsymbol{a}$ denotes the particle acceleration, $\boldsymbol{r}$ the position vector at the field point with respect to q. The right handed side is evaluated at the retarded time.

(i) Find the power radiated per unit solid angle $dP/d\Omega$ into the direction $\hat{\boldsymbol{r}}$ in terms of a and χ, where χ is the angle between $\hat{\boldsymbol{r}}$ and $\boldsymbol{a}$.

(ii) An incoming harmonic electromagnetic wave is described by $\boldsymbol{E}_{\text{in}} = E_0\hat{\boldsymbol{x}}\cos(kz - \omega t)$.
Find the time-averaged energy flux I, the rate of the energy transmitted per unit area across the xy plane, in terms of the input amplitude E_0, k, ω.

(iii The electric field of this wave drives a nano-particle of mass m and charge q to oscillate around the origin on the xy plane at $z = 0$. Since the motion is highly non-relativistic, we can neglect the magnetic force and the self-reaction. The oscillation generates secondary radiation into all possible directions, specified by the polar angle θ with respect to the z axis and the azimuthal angle ϕ of the spherical coordinates.

Determine the *time-averaged* power per unit solid angle, $\frac{dP}{d\Omega}$, emitted from the oscillating particle, in terms of $q, m, E_0, k, \omega, \theta, \phi$.

Into which direction, does $\frac{dP}{d\Omega}$ vanish?

(iv) Find the total cross-section $\sigma \equiv (\int \frac{dP}{d\Omega} d\Omega)/I$.

$\boldsymbol{E}_{\text{rad}}$ is perpendicular to $\hat{\boldsymbol{r}}$.

$$\boldsymbol{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0 c^2} \frac{1}{r} \left[(\hat{\boldsymbol{r}} \cdot \boldsymbol{a})\hat{\boldsymbol{r}} - \boldsymbol{a} \right] \ .$$

Note that the acceleration projection a_r along $\hat{\boldsymbol{r}}$ is removed from $\boldsymbol{a}$ for the contribution in $\boldsymbol{E}_{\text{rad}}$.

The Poynting vector is along $\hat{\boldsymbol{r}}$.

$$\boldsymbol{S} = c\epsilon_0 E_{\text{rad}}^2 \hat{\boldsymbol{r}} = c\epsilon_0 \left(\frac{qa}{4\pi\epsilon_0 c^2} \right)^2 \frac{\sin^2 \chi}{r^2} \ \hat{\boldsymbol{r}} \ .$$

$$\frac{dP}{d\Omega} = r^2 S = \frac{q^2 a^2 \sin^2 \chi}{16\pi^2 \epsilon_0 c^3} \ .$$

The time-averaged energy flux $I = \frac{1}{2} c\epsilon_0 E_0^2$.

The Newton's 2nd Law gives the acceleration of the nano-particle q to be $\boldsymbol{a} = q\boldsymbol{E}_{\text{in}}/m$. Also, $\cos^2 \chi = (\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{r}})^2 = \sin^2 \theta \cos^2 \phi$

$$\boxed{\frac{dP}{d\Omega} = \frac{q^4 E_0^2}{32\pi^2 \epsilon_0 m^2 c^3} (1 - \sin^2 \theta \cos^2 \phi) \ .}$$

The radiation vanishes at $\theta = \frac{\pi}{2}, \phi = 0$.

$$\frac{d\sigma}{d\Omega} \equiv \frac{1}{I} \frac{dP}{d\Omega} = \left(\frac{q^2}{4\pi\epsilon_0 mc^2} \right)^2 (1 - \sin^2 \theta \cos^2 \phi) \ .$$

$$\boxed{\sigma = \frac{8\pi}{3} \left(\frac{q^2}{4\pi\epsilon_0 mc^2} \right)^2 \ .}$$

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5. An electrostatic potential near the origin of an observer $\mathcal{O}$ is given by $V(x,y,z) = (V_0/a^2)(y^2+z^2)$. The absence of magnetic field implies the corresponding vector potential $\boldsymbol{A} = 0$. Find the charge density ρ near the origin (in terms of input parameters V_0, a, ϵ_0).

An observer $\mathcal{O}'$ slides with velocity v along the overlapping x, x' axes. The other axis pairs are parallel, $y \parallel y'$, $z \parallel z'$. As $(V/c, \boldsymbol{A})$ transforms into $(V'/c, \boldsymbol{A}')$ like $(x^0, \boldsymbol{x})$ into $(x'^0, \boldsymbol{x}')$, determine V' and $\boldsymbol{A}'$.

Find the electric and magnetic fields observed by $\mathcal{O}'$.

Find the corresponding charge density ρ', and the current density $\boldsymbol{J}'$.

$$\boldsymbol{E} = -\boldsymbol{\nabla} V = -(V_0/a^2)(2y\hat{\boldsymbol{y}} + 2z\hat{\boldsymbol{z}}) \ , \ \boldsymbol{\nabla}\cdot\boldsymbol{E} = -4V_0/a^2 \ , \ \rho = -4\epsilon_0 V_0/a^2 \ .$$

The Lorentz transformation gives

$$A'^x = -\gamma v V/c^2 = -\gamma v (V_0/a)(y'^2 + z'^2)/c^2 \ , \ A'^y = 0 \ , \quad A'^z = 0 \ , V' = \gamma V \ .$$

$$\boldsymbol{E}' = -\tfrac{\partial}{\partial t'}\boldsymbol{A}' - \boldsymbol{\nabla}' V' = \gamma \boldsymbol{E} \ .$$

$$\boldsymbol{B}' = \boldsymbol{\nabla}' \times \boldsymbol{A}' = -\frac{\gamma v V_0}{c^2 a^2}\begin{vmatrix} \hat{\boldsymbol{x}}' & \hat{\boldsymbol{y}}' & \hat{\boldsymbol{z}}' \\ \partial_{x'} & \partial_{y'} & \partial_{z'} \\ y'^2 + z'^2 & 0 & 0 \end{vmatrix} = -2\frac{\gamma v V_0}{c^2 a^2}(z'\hat{\boldsymbol{y}}' - y'\hat{\boldsymbol{z}}')$$

$$\boldsymbol{\nabla}' \times \boldsymbol{B}' = 4\frac{\gamma v V_0}{c^2 a^2}\hat{\boldsymbol{x}}' = \mu_0 \boldsymbol{J}' \ .$$

So, $\boxed{\boldsymbol{J}' = 4\gamma\epsilon_0 V_0 \boldsymbol{v}/a^2.}$ Also, div$'$ $\boldsymbol{E}'$ gives $\boxed{\rho = \gamma\rho'.}$ Thus, $\boldsymbol{J}' = \rho'\boldsymbol{v}$ as expected. We notice that $\boldsymbol{J}'$ can also be obtained from Lorentz transformation of the 4-current directly.

Formulas

$$\oint \boldsymbol{B}\cdot d\boldsymbol{\ell} = \mu_0(I_{q,\text{enc}} + I_{\text{disp}}) \ , \quad I_{\text{disp}} = \epsilon_0 \tfrac{d}{dt}\int d^2\boldsymbol{a}\cdot\boldsymbol{E} \ .$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \ , \quad \boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho/\epsilon_0 \ ,$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\tfrac{\partial}{\partial t}\boldsymbol{B} \ , \qquad \boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0(\boldsymbol{J} + \epsilon_0\tfrac{\partial}{\partial t}\boldsymbol{E}) \ .$$

$$\boldsymbol{f} = \rho\boldsymbol{E} + \boldsymbol{J}\times\boldsymbol{B} \ , \quad \boldsymbol{E} = -\tfrac{\partial}{\partial t}\boldsymbol{A} - \boldsymbol{\nabla}V \ , \quad \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \ .$$

Solid angle: $\Omega = 2\pi(1-\cos\theta)$.
Biot-Savart Law:

$$\boldsymbol{B}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{x'})\times(\boldsymbol{x}-\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|^3} \longrightarrow \frac{\mu_0}{4\pi}\int \frac{d\boldsymbol{\ell}\times(\boldsymbol{x}-\boldsymbol{x_\ell})}{|\boldsymbol{x}-\boldsymbol{x_\ell}|^3}$$

The current I in a line segment gives

$$B = \frac{\mu_0 I}{4\pi r}(\cos\theta_{\text{end}} - \cos\theta_{\text{front}}) \ ,$$

at the point P with the front/end angle $\theta_{\text{front/end}}$. A circular current loop of radius R gives $B = \frac{\mu_0 I}{2R}$ at its center. In general, for a magnetic dipole,

$$\boldsymbol{B}(r,\theta) = \frac{\mu_0 m}{4\pi}\left(\frac{2\cos\theta}{r^3}\hat{\boldsymbol{r}} + \frac{\sin\theta}{r^3}\hat{\boldsymbol{\theta}}\right) \ ,$$

$$u_{\text{em}} = \tfrac{1}{2}(\epsilon_0 E^2 + \tfrac{1}{\mu_0}B^2) \ , \quad \boldsymbol{S} = \tfrac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} \ , \ \omega = vk \ , \ v = c/n \ .$$

Lorentz Transformation:

$$x' = \gamma(x - vt) \ , \ y' = y \ , \ z' = z \ , \ t' = \gamma(t - xv/c^2) \ , \ x^0 = ct \ , \ \gamma = 1/\sqrt{1 - v^2/c^2} \ .$$