University of Illinois at Chicago
Department of Physics

Electricity and Magnetism
PhD Qualifying Examination

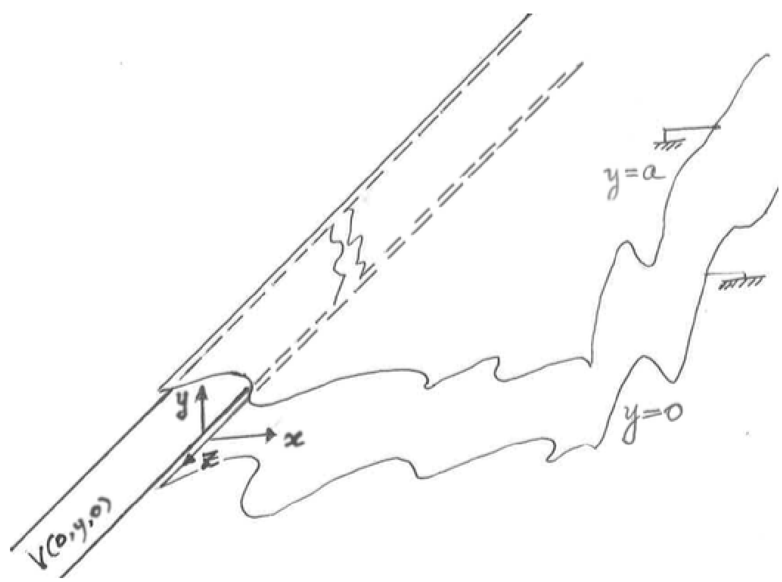
January 8, 2016 (Friday)
9:00 am - 12:00 noon

Full credit can be achieved from completely correct answers to **4 questions**. If the student attempts all 5 questions, all of the answers will be graded, and the **top 4 scores** will be counted toward the exam's total score.

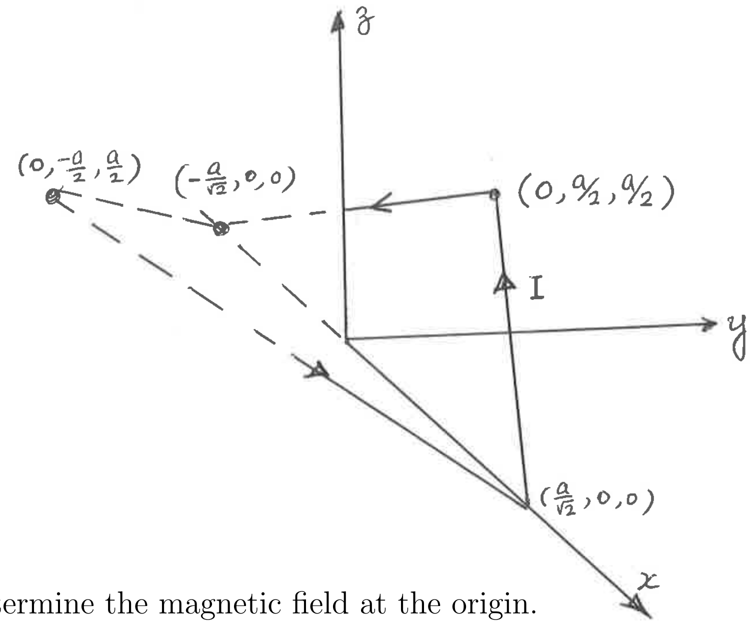
1. Two semi-infinite grounded metal plates are parallel to the xz plane, one at $y = 0$, the other at $y = a$. The left end of their gap, at $x = 0$, is capped off with an infinitely long strip insulated from the two plates and maintained at a specific z-independent potential $V(0, y, z)$. By the method of separation of variables, the potential inside this "slot" $(x > 0)$ is known to be

$$V(x, y, z) = \sum_{n=1}^{\infty} C_n e^{-k_n x/a} \sin(n\pi y/a).$$

- Explicitly find the exponent coefficient k_n in terms of a and n based on the Laplace equation.
- Now if the left end potential is explicitly set up to be $V(0, y, z) = V_0 \sin(7\pi y/a)$, determine all C_n *simply by inspection*. Explicitly find the charge residing in the area between $z = 0$ and $z = b$ on the lower xz plane $(y = 0)$, in terms of V_0, a, b, ϵ_0.
- Explicitly determine the resultant electrostatic force acting on the above mentioned area.



2. A square circuit $a \times a$ is folded along the diagonal line into two perpendicular isosceles right triangles. The fold-line is placed along the x axis. The center of the circuit is placed at the origin O. The current I flows around the first isosceles, starting from $(a/\sqrt{2}, 0, 0)$, reaching $(0, a/2, a/2)$, then $(-a/\sqrt{2}, 0, 0)$. The current moves around the second isosceles, flowing to $(0, -a/2, a/2)$, and returning to the beginning point.



- Determine the magnetic field at the origin.
- Determine the *leading* dipole magnetic field $\boldsymbol{B}$ at a large distance $r \gg a$.
- A secondary circular loop is placed at the spherical coordinates of fixed $r = R \gg a$, $\theta = 60^\circ$ and $\phi \in [0, 2\pi]$. Find the leading dipole contribution of the mutual inductance between the two circuits.
- If the secondary loop carries another current I', determine the leading contribution of the magnetic flux due to I' through the first small circuit.

3. (i) Determine the electric flux of the Coulomb field from a point charge q located at $(0, 0, z)$ through a circular disk of radius R, on the xy plane and centered at $(0, 0, 0)$ with its normal $\hat{\boldsymbol{n}} = +\hat{\boldsymbol{z}}$. Show the result $\Phi_e \equiv \int_{\text{disk}} \boldsymbol{E} \cdot \hat{\boldsymbol{n}} d^2a$ explictly in terms of q, R, z. Describe any discontinuity.

   (ii) The point charge q moves at a uniform velocity v along the z axis. Its position is $z = vt$. In the non-relativistic limit for the slow motion, the corresponding electric field is simply given by the instantaneous Coulomb form.

   Determine the Maxwell displacement current I_d through the disk area, in terms of q, R, v, t. Use appropriate mathematical prescription for the singular behavior in the interval when $t \approx 0$.

   (iii) Summing up the displacement current I_d and the particle q current $I_q = qv\delta(vt)$, find the Amperian integral $\oint \boldsymbol{B} \cdot d\boldsymbol{\ell}$, in terms of q, R, v, t.

   (iv) Explicitly give the magnetic field at $(R, 0, 0)$ based on above reasoning and the symmetry, for all t.

4. An accelerated charged particle q radiates the retarded electric field, approximately

given by

$$\boldsymbol{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0 c^2}\left(\frac{\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \boldsymbol{a})}{r}\right) ,$$

where $\boldsymbol{a}$ denotes the particle acceleration, $\boldsymbol{r}$ the position vector at the field point with respect to q. The right handed side is evaluated at the retarded time.

(i) Find the power radiated per unit solid angle $dP/d\Omega$ into the direction $\hat{\boldsymbol{r}}$ in terms of a and χ, where χ is the angle between $\hat{\boldsymbol{r}}$ and $\boldsymbol{a}$.

(ii) An incoming harmonic electromagnetic wave is described by $\boldsymbol{E}_{\text{in}} = E_0 \hat{\boldsymbol{x}} \cos(kz - \omega t)$.
Find the time-averaged energy flux I, the rate of the energy transmitted per unit area across the xy plane, in terms of the input amplitude E_0, k, ω.

(iii The electric field of this wave drives a nano-particle of mass m and charge q to oscillate around the origin on the xy plane at $z = 0$. Since the motion is highly non-relativistic, we can neglect the magnetic force and the self-reaction. The oscillation generates secondary radiation into all possible directions, specified by the polar angle θ with respect to the z axis and the azimuthal angle ϕ of the spherical coordinates.

Determine the *time-averaged* power per unit solid angle, $\frac{dP}{d\Omega}$, emitted from the oscillating particle, in terms of $q, m, E_0, k, \omega, \theta, \phi$.

Into which direction, does $\frac{dP}{d\Omega}$ vanish?

(iv) Find the total cross-section $\sigma \equiv (\int \frac{dP}{d\Omega} d\Omega)/I$.

5. An electrostatic potential near the origin of an observer $\mathcal{O}$ is given by $V(x, y, z) = (V_0/a^2)(y^2 + z^2)$. The absence of magnetic field implies the corresponding vector potential $\boldsymbol{A} = 0$. Find the charge density ρ near the origin (in terms of input parameters V_0, a, ϵ_0).

An observer $\mathcal{O}'$ slides with velocity v along the overlapping x, x' axes. The other axis pairs are parallel, $y \parallel y'$, $z \parallel z'$. As $(V/c, \boldsymbol{A})$ transforms into $(V'/c, \boldsymbol{A}')$ like $(x^0, \boldsymbol{x})$ into $(x'^0, \boldsymbol{x}')$, determine V' and $\boldsymbol{A}'$.

Find the electric and magnetic fields observed by $\mathcal{O}'$.

Find the corresponding charge density ρ', and the current density $\boldsymbol{J}'$.

Formulas

$$\oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_0 (I_{q,\text{enc}} + I_{\text{disp}}) \ , \quad I_{\text{disp}} = \epsilon_0 \tfrac{d}{dt} \int d^2\boldsymbol{a} \cdot \boldsymbol{E} \ .$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \ , \quad \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\epsilon_0 \ ,$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\tfrac{\partial}{\partial t}\boldsymbol{B} \ , \qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 (\boldsymbol{J} + \epsilon_0 \tfrac{\partial}{\partial t}\boldsymbol{E}) \ .$$

$$\boldsymbol{f} = \rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} \ , \quad \boldsymbol{E} = -\tfrac{\partial}{\partial t}\boldsymbol{A} - \boldsymbol{\nabla} V \ , \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \ .$$

Solid angle: $\Omega = 2\pi(1 - \cos\theta)$.
Biot-Savart Law:

$$\boldsymbol{B}(\boldsymbol{x}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{x'}) \times (\boldsymbol{x} - \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|^3} \longrightarrow \frac{\mu_0}{4\pi} \int \frac{d\boldsymbol{\ell} \times (\boldsymbol{x} - \boldsymbol{x_\ell})}{|\boldsymbol{x} - \boldsymbol{x_\ell}|^3}$$

The current I in a line segment gives

$$B = \frac{\mu_0 I}{4\pi r} (\cos\theta_{\text{end}} - \cos\theta_{\text{front}}) \ ,$$

at the point P with the front/end angle $\theta_{\text{front/end}}$. A circular current loop of radius R gives $B = \frac{\mu_0 I}{2R}$ at its center. In general, for a magnetic dipole,

$$\boldsymbol{B}(r, \theta) = \frac{\mu_0 m}{4\pi} \left(\frac{2\cos\theta}{r^3} \hat{\boldsymbol{r}} + \frac{\sin\theta}{r^3} \hat{\boldsymbol{\theta}} \right) \ ,$$

$$u_{\text{em}} = \tfrac{1}{2}(\epsilon_0 E^2 + \tfrac{1}{\mu_0} B^2) \ , \quad \boldsymbol{S} = \tfrac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B} \ , \ \omega = vk \ , \ v = c/n \ .$$

Lorentz Transformation:

$$x' = \gamma(x - vt) \ , \ y' = y \ , \ z' = z \ , \ t' = \gamma(t - xv/c^2) \ , \ x^0 = ct \ , \ \gamma = 1/\sqrt{1 - v^2/c^2} \ .$$