Solutions

1. (a) From the Lorentz force $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$, we have $m\ddot{\boldsymbol{r}} = m\dot{\boldsymbol{v}} = qq_M \frac{\boldsymbol{v}\times\boldsymbol{r}}{2r^3}$.

   (b) $\boldsymbol{L} \equiv m\boldsymbol{r} \times \boldsymbol{v}$, and

$$\dot{\boldsymbol{L}} = m\boldsymbol{r} \times \dot{\boldsymbol{v}} = qq_M \boldsymbol{r} \times \frac{(\boldsymbol{v} \times \boldsymbol{r})}{2r^3} = qq_M \frac{r^2\boldsymbol{v} - (\boldsymbol{r}\cdot\boldsymbol{v})\boldsymbol{r}}{2r^3} .$$

   This is not zero in general. On the other hand,

$$\frac{d}{dt}\left(\frac{\boldsymbol{r}}{r}\right) = \frac{\boldsymbol{v}}{r} + \boldsymbol{r}\left(-\frac{1}{r^2}\right)\dot{r} = \frac{\boldsymbol{v}}{r} - \frac{\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{v})}{r^3} ,$$

   where $\dot{r} = \frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r}$. From these, we have $\dot{\boldsymbol{L}}_0 = 0$ for $C = -qq_M/2$.

   (c) Since $\boldsymbol{v}\cdot(\boldsymbol{v}\times\boldsymbol{r}) = 0$, we have $\boldsymbol{v}\cdot\boldsymbol{F} = 0$ and the rate of change of the kinetic energy $\dot{K} = m\boldsymbol{v}\cdot\dot{\boldsymbol{v}} = \boldsymbol{v}\cdot\boldsymbol{F} = 0$.

   (d) Taking $\boldsymbol{L}_0\cdot\boldsymbol{r} = rL_0\cos\theta$ on both sides of the definition of $\boldsymbol{L}_0$, and since $\boldsymbol{L}\cdot\boldsymbol{r} = 0$, we have $rL_0\cos\theta = Cr$, which gives $\cos\theta = C/L_0$, which is constant in time. From the figure, working out in components, $\boldsymbol{L}_0 = L_0\cos\theta\hat{\boldsymbol{r}} - L_0\sin\theta\hat{\boldsymbol{\theta}}$ and

$$\boldsymbol{L} = m\boldsymbol{r}\times\boldsymbol{v} = mr\hat{\boldsymbol{r}} \times (\dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\dot{\phi}\hat{\boldsymbol{\phi}}) = mr^2(\dot{\theta}\hat{\boldsymbol{\phi}} - \sin\theta\dot{\phi}\hat{\boldsymbol{\theta}}) ,$$

   which gives $\boldsymbol{L}_0 = mr^2(\dot{\theta}\hat{\boldsymbol{\phi}} - \sin\theta\dot{\phi}\hat{\boldsymbol{\theta}}) + C\hat{\boldsymbol{r}}$. Comparing the two expressions for $\boldsymbol{L}_0$ we reproduce $\dot{\theta} = 0$, $\cos\theta = C/L_0$ and finally obtain $\dot{\phi} = L_0/(mr^2)$. The conserved kinetic energy is

$$K = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2) = \frac{1}{2}m\dot{r}^2 + \frac{\sin^2\theta L_0^2}{2mr^2} ,$$

   which gives the effective radial potential $V_{\text{eff}}(r) = \frac{\sin^2\theta L_0^2}{2mr^2}$, which is a monotonically decreasing function on r. The minimum radius $r_{\min}$ is determined by solving $E = \frac{\sin^2\theta L_0^2}{2mr_{\min}^2}$ where E is the conserved kinetic energy.

2. (a) The equations of motion are

$$2m\ddot{x}_1 = -2kx_1 + k(x_2 - x_1) = -3kx_1 + kx_2\,, \quad m\ddot{x}_2 = -k(x_2 - x_1) = kx_1 - kx_2\,,$$

which can be written in a matrix form

$$\begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -3k & k \\ k & -k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(b) To find the normal mode frequency ω, we use the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-i\omega t} \begin{pmatrix} a \\ b \end{pmatrix},$$

for some constants a and b, and inserting this into the equation of motion, we have

$$\begin{pmatrix} 2m\omega^2 - 3k & k \\ k & m\omega^2 - k \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0\,.$$

To have a non-zero normal mode of a, b, we have the vanishing determinant

$$(2m\omega^2 - 3k)(m\omega^2 - k) - k^2 = 2(m\omega^2)^2 - 5k(m\omega) + 2k^2 = (2m\omega^2 - k)(m\omega^2 - 2k) = 0\,,$$

whose solution gives $\omega_1 = \sqrt{2k/m}$ and $\omega_2 = \sqrt{k/(2m)}$.

(c) The new equations of motion in a matrix form is

$$\begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -3k & k \\ k & -k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}.$$

Assuming the same form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-i\omega t} \begin{pmatrix} a \\ b \end{pmatrix},$$

we have

$$\begin{pmatrix} 2m\omega^2 - 3k & k \\ k & m\omega^2 - k + i\gamma\omega \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0\,,$$

which gives the equation

$$(2m\omega^2 - 3k)(m\omega^2 - k + i\gamma\omega) - k^2 = (2m\omega^2 - k)(m\omega^2 - 2k) + i\gamma\omega(2m\omega^2 - 3k) = 0\,,$$

which we solve in linear perturbation theory in γ.

(d) The imaginary part of ω represents exponential decay (damping) of the motions in time.

3. (a) The total energy is $E = (M + m\gamma)c^2$ where $\gamma \equiv 1/\sqrt{1-(v/c)^2}$, and the total momentum is $P = mv\gamma$. Since $E^2 - P^2c^2$ is Lorentz invariant, the energy in the center of mass frame E' is given by

$$E' = \sqrt{E^2 - P^2c^2} = c\sqrt{(M+m\gamma)^2c^2 - m^2v^2\gamma^2} = c^2\sqrt{M^2 + 2mM\gamma + m^2}\,.$$

After the fission, half of E' is carried by one of the daughter nuclei with momentum p' whose energy is $c\sqrt{(p')^2 + (M')^2c^2}$. Equating the two gives

$$p' = \frac{c}{2}\sqrt{M^2 + 2mM\gamma + m^2 - 4(M')^2}\,.$$

(b) Let the magnitude of the momentum of either e^- or $\bar{\nu}_e$ be p_2. From momentum conservation, we have $2p_2\cos\theta = p_1$ and energy conservation gives $2p_2c + c\sqrt{p_1^2 + m_p^2c^2} = m_nc^2$. Eliminating p_2 from the first equation, and using the second, we get

$$\cos\theta = \frac{p_1}{m_nc - \sqrt{(p_1)^2 + m_p^2c^2}}\,.$$

The expression on the right-hand side is a monotonically increasing function of p_1. Since $0 \le \cos\theta \le 1$, we have

$$0 \le p_1 \le \frac{(m_n^2 - m_p^2)c}{2m_N}\,.$$

When $p_1 = 0$, we have $\theta = \pi/2$ and e^- and $\bar{\nu}_e$ are back to back. In the other limit when $p_1 = (m_n^2 - m_p^2)c/(2m_N)$ we have $\theta = 0$ and $(e^-, \bar{\nu}_e)$ pair and the proton are back to back.

4. (a) The principle axes are symmetric axis of the top and the two perpendicular, mutually orthogonal directions. The I_3 around the symmetric axis is computed as

$$I_3 = \frac{M}{\pi R^2} \int_0^R dr\, 2\pi r \cdot r^2 = \frac{MR^2}{2} \,.$$

Using perpendicular axis theorem and the parallel axis theorem, the other moment of inertia are $I_1 = I_2 \equiv I = I_3/2 + Ml^2 = MR^2/4 + Ml^2$. $\boldsymbol{I} = \mathrm{diag}(I, I, I_3)$.

(b) In $(\hat{\mathbf{1}}, \hat{\mathbf{3}})$ basis which are principal axes, we have $\omega_1 = \omega_0 \sin\theta$ and $\omega_3 = S + \omega_0 \cos\theta$, which gives the angular momentum in this basis as $L_1 = I\omega_0 \sin\theta$ and $L_3 = I_3(S + \omega_0 \cos\theta)$. Using the relation to $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ basis

$$\hat{\mathbf{1}} = -\cos\theta \hat{\boldsymbol{x}} + \sin\theta \hat{\boldsymbol{z}} \,, \quad \hat{\mathbf{3}} = \sin\theta \hat{\boldsymbol{x}} + \cos\theta \hat{\boldsymbol{z}} \,,$$

we have $L_x = I_3 S \sin\theta + (I_3 - I)\omega_0 \sin\theta \cos\theta$, and $L_z = I_3(S + \omega_0 \cos\theta)\cos\theta + I\omega_0 \sin^2\theta$.

(c) The magnitude of the torque is $\tau = lMg\sin\theta$, and it points inward the paper. The magnitude of the $d\boldsymbol{L}/dt$ is $L_x \omega_0$ to the same direction. Equating the two gives $lMg\sin\theta = (I_3 S \sin\theta + (I_3 - I)\omega_0 \sin\theta\cos\theta)\omega_0$ whose solution is

$$\omega_0 = \frac{-I_3 S + \sqrt{(I_3 S)^2 + 4lMg(I_3 - I)\cos\theta}}{2(I_3 - I)\cos\theta} \,.$$

When $S^2 \gg gl/R^2$, the first term inside square root is much larger than the second term, and performing Taylor expansion, we get $\omega_0 \approx lMg/(I_3 S)$. The result is independent of the angle θ.

(d) To find the force, we simply consider the motion of the center of mass. To balance the gravity, there should be a normal force of magnitude $N = Mg$. The center of mass undergoes a rotational motion with radius $l\sin\theta$ and angular frequency ω_0, so the necessary centripetal force coming from the surface friction is

$$f = M(l\sin\theta)\omega_0^2 = \frac{l^3 M^3 g^2 \sin\theta}{I_3^2 S^2} \,.$$

Since $f \le \mu_s N$, we have $\mu_s \ge l^3 M^2 g \sin\theta / (I_3^2 S^2)$.

5. (a) Taking time derivatives, we have

$$\dot{x} = R(1 + \cos\phi)\dot{\phi}\,, \quad \dot{y} = R(\sin\phi)\dot{\phi}\,,$$

so that the Lagrangian is

$$\begin{aligned} L &= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - mgy = mR^2(1 + \cos\phi)\dot{\phi}^2 - mgR(1 - \cos\phi) \\ &= 2mR^2\cos^2(\phi/2)\dot{\phi}^2 - 2mgR\sin^2(\phi/2)\,. \end{aligned}$$

(b) From $s = 4R\sin(\phi/2)$, taking time derivative we have $\dot{s} = 2R\cos(\phi/2)\dot{\phi}$. It is straightforward to see that the Lagrangian in terms of s becomes

$$L = \frac{m}{2}\dot{s}^2 - \frac{mg}{8R}s^2\,.$$

(c) The Lagrange equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0\,,$$

is given by $m\ddot{s} + (mg/4R)s = 0$, which is a simple harmonic motion. The solution is then

$$s(t) = A\cos(\omega t + \theta_0)\,, \quad \omega = \sqrt{\frac{g}{4R}}\,,$$

for any constants A and θ_0. The period is $T = (2\pi)/\omega = (2\pi)\sqrt{4R/g}$.