Classical Mechanics

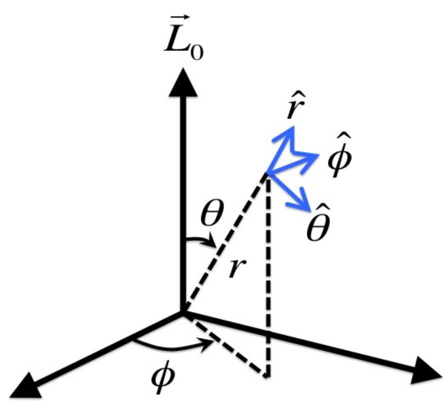
1. Let's consider a particle of charge q with mass m moving in the presence of a magnetic field
$$\boldsymbol{B} = q_M \frac{\boldsymbol{r}}{2r^3} ,$$
where $\boldsymbol{r}$ is the 3 dimensional position vector, and $r \equiv |\boldsymbol{r}|$, and q_M is a constant.

   (a) Write down the equation of motion.

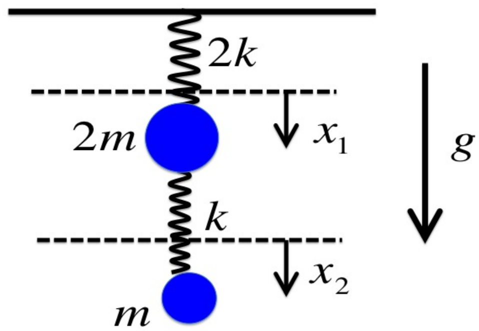
   (b) Using the equation of motion, show that the following is a conserved quantity
$$\boldsymbol{L}_0 \equiv m\boldsymbol{r} \times \boldsymbol{v} + C\frac{\boldsymbol{r}}{r} ,$$
for some constant C, and find C. $\boldsymbol{v} \equiv \frac{d\boldsymbol{r}}{dt}$ is the velocity. You may find the following identity useful: for three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$,
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}) .$$

   (c) Show that the kinetic energy $E = \frac{1}{2}m\boldsymbol{v}^2$ is a constant of motion.

   (d) Let $\boldsymbol{L}_0$ point to the $\hat{\boldsymbol{z}}$ direction, and let's introduce spherical coordinate system (r, θ, ϕ) as in the figure. From
$$\boldsymbol{r} = r\hat{\boldsymbol{r}} , \quad \boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\dot{\phi}\hat{\boldsymbol{\phi}} ,$$
and the result of (b), show that θ is a constant of motion, and
$$\dot{\phi} = \frac{L_0}{mr^2} , \quad L_0 \equiv |\boldsymbol{L}_0| .$$
Express θ in terms of L_0 and C. Also find the effective potential for the radial motion, $V_{\text{eff}}(r)$. Is there minimum radius $r_{\min}$ for a given L_0 and E?



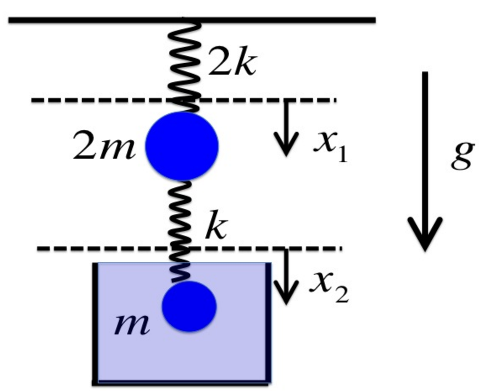
2. Consider motions of the two masses, $2m$ and m, attached to the two springs of spring constants $2k$ and k as in the figure. Let the constant acceleration of gravity be g. Denote the displacement of the each mass from its equilibrium position as x_1 and x_2 respectively.



(a) Write down the equations of motion for x_1 and x_2, and express them in a matrix form.

(b) Assuming the time dependence of the form $e^{-i\omega t}$ with normal mode frequency ω, find the two possible normal mode frequencies of positive sign.

(c) The second mass m is now immersed in water bottle, so that there is a frictional force $f = -\gamma \dot{x}_2$. See the figure below. Write down the equations of motion in a matrix form, and write down the equation that determines complex valued normal mode frequencies (you do NOT need to solve the equation).



(d) Explain qualitatively the meaning of the imaginary part of the normal mode frequency.

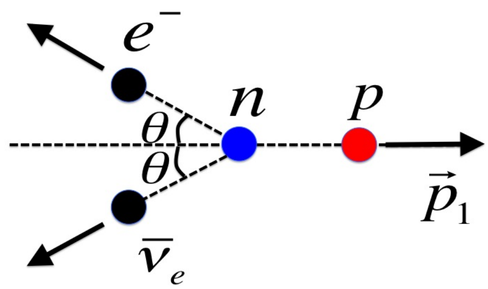
3. The following two problems are independent and should be solved using special theory of relativity.

   (a) A heavy nucleus of mass M staying at rest absorbs a relativistically fast neutron of mass m and velocity v, and then undergoes a fission process into two identical daughter nuclei of rest mass M'. The final velocities of the two daughter nuclei are collinear to the initial velocity of the neutron. Find the magnitude of the momenta of the two daughter nuclei in the center of mass frame of the pair.

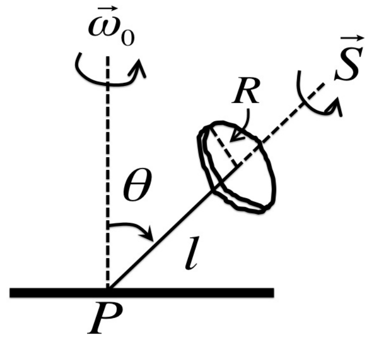
   (b) A neutron at rest in the lab frame undergoes a β-decay

$$n \longrightarrow p + e^- + \bar{\nu}_e \,.$$

   Let's assume that the electron e^- and neutrino $\bar{\nu}_e$ are massless, and their final momenta have the same magnitude, but make an angle 2θ, as in the figure. Express the angle θ in terms of neutron mass m_n, proton mass m_p, and the magnitude of the final proton momentum p_1. From this, obtain the minimum and maximum values of p_1, and the corresponding angles θ.



4. A rotating top consists of a homogeneous disc with mass M and radius R attached to one end of a massless rigid rod of length l about its symmetry axis as shown in the figure. The other end, the pivot point P, is held fixed by the surface friction with a static friction coefficient μ_s. The top is rotating with constant angular speed S around its symmetry axis, and at the same time is precessing around the vertical direction with angular speed ω_0. The angle θ from the vertical direction is constant in time. Let the constant acceleration of gravity be g.

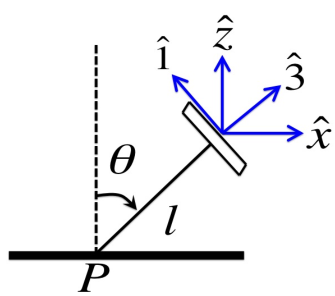


(a) Find the 3×3 moment of inertia tensor $\boldsymbol{I}$ of the top about the pivot point P, in the basis which diagonalizes $\boldsymbol{I}$.

(b) Let's introduce the unit vectors $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ and $(\hat{\mathbf{1}}, \hat{\mathbf{3}})$ as in the figure, so that we can write the total angular velocity as

$$\boldsymbol{\omega} = S\,\hat{\mathbf{3}} + \omega_0\,\hat{\boldsymbol{z}}\,.$$

Find the angular momentum $\boldsymbol{L}$ about the pivot point P in the $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ basis. Use ω_0, S, θ as given data.



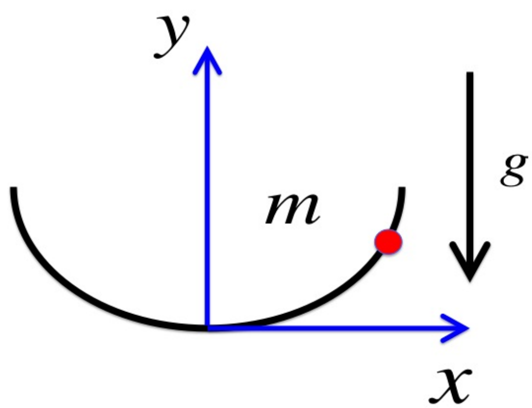
(c) Find the torque $\boldsymbol{\tau}$ about the pivot point P, and from the equation of motion $d\boldsymbol{L}/dt = \boldsymbol{\tau}$, express ω_0 in terms of M, R, S, l, g and θ. Show that in the limit $S^2 \gg gl/R^2$ or equivalently $S \gg \omega_0$, we have $\omega_0 \approx lMg/(I_3 S)$.

(d) Using the result of (c) when $S^2 \gg gl/R^2$, that is $\omega_0 \approx lMg/(I_3 S)$, find the force $\boldsymbol{F}$ acting on the pivot point P, and from this, find the condition on the friction constant μ_s for the surface to be able to sustain the configuration. To find the force, focus only on the motion of the center of mass.

5. A bead of mass m slides along a frictionless curve parametrized by

$$x(\phi) = R(\phi + \sin\phi) , \quad y(\phi) = R(1 - \cos\phi) ,$$

where ϕ is a parameter of the curve, and $R > 0$ is a constant, and (x, y) is a Cartesian coordinate system as shown in the figure. Let the constant acceleration of gravity be g pointing to the negative y direction, and $\phi = 0$ corresponds to the stable minimum point $(x, y) = (0, 0)$.



(a) Write down the Lagrangian of the particle in terms of the generalized coordinate ϕ for the particle which is chosen to be the same ϕ parameterizing the curve.

(b) Using the change of variable from ϕ to $s \equiv 4R\sin(\phi/2)$, write down the Lagrangian in terms of the new variable s. You may find the following identities useful:

$$1 - \cos\phi = 2\sin^2(\phi/2) , \quad 1 + \cos\phi = 2\cos^2(\phi/2) .$$

(c) Write down the Lagrange equation of motion for s, and solve it to find the period of small amplitude oscillatory motions.