University of Illinois at Chicago
Department of Physics

Thermodynamics & Statistical Mechanics
PhD Qualifying Examination

January 5, 2015
9.00 am – 12:00 pm

Full credit can be achieved from completely correct answers to <u>4 questions</u>. If the student attempts all 5 questions, all of the answers will be graded, and the <u>top 4 scores</u> will be counted toward the exam's total score.

Problem 1

Consider a classical canonical system in one dimension that is described by the Hamiltonian

$$H = c_1 \, |q|^{\alpha_1}$$

with q being a generalized cartesian coordinate, and c_1,$\alpha_1 > 0$.

a) Compute the partition function of the particle.

b) Compute the free energy F, the total internal energy, U, and the entropy S of the system.

c) Consider next a classical system in two dimensions that is described by the Hamiltonian

$$H = c_1 \, |q_1|^{\alpha_1} + c_2 \, |q_2|^{\alpha_2}$$

with $q_{1,2}$ being independent, generalized, cartesian coordinates. Under which conditions for the parameters c_1, c_2, α_1, and α_2 does the equipartition theorem hold? Explain your result.

Problem 2

Consider two coupled quantum mechanical systems with Hamiltonian

$$H = \hbar\omega_0 n + \hbar\omega_0 m + \alpha\hbar\omega_0 nm$$

where $n, m = 0, 1$ can take only the values 0,1.

a) Compute the partition function of the system.

b) Compute the free energy of the system. Expand the free energy to leading order in

$$\alpha\hbar\omega_0/(k_B T) \ll 1$$

c) Compute the entropy of the system.

d) Is the entropy increased or decreased by the coupling-term with $\alpha > 0$? Explain your result.

Problem 3

a) Blackbody radiation can be treated as a macroscopic thermodynamic system. Its energy density if given by

$$U = \frac{4}{c} V \sigma T^4$$

where σ is the Stefan-Boltzmann constant. Determine the form of the fundamental relation whose independent variables are V amd T, and obtain expressions for the pressure and specific heat (note that the entropy S for the systems vanishes at $T = 0$).

b) Starting from the entropy

$$S = N k_B \left[\frac{5}{2} + \ln \left\{ \frac{V}{N} \left(\frac{4\pi m}{3h^2} \frac{U}{N} \right)^{3/2} \right\} \right]$$

and internal energy

$$U = \frac{3}{2} N k_B T$$

of the ideal gas, compute the system's free energy and its grandcanonical potential.

Problem 4

Consider the earth's atmosphere as an ideal gas of with molecular weight μ in a gravitational field with g being the acceleration due to gravity.

a) If z denotes the height above sea level, show that the change in the atmospheric pressure p with height is given by

$$\frac{dp}{p} = -\frac{\mu g}{N_A k_B T} dz$$

where T is the temperature at height z.

b) If the decrease in pressure is due to an adiabatic expansion, i.e., using

$$pV^\gamma = const.$$

show that

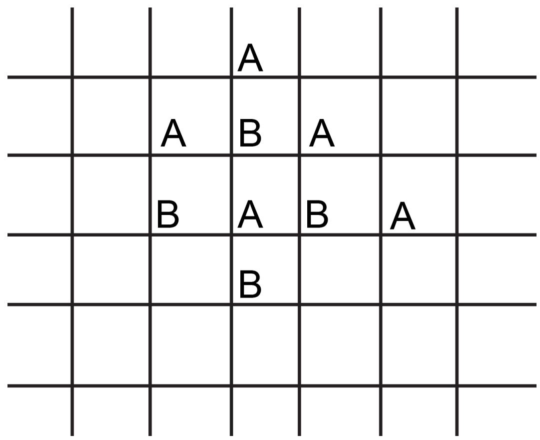
$$\frac{dp}{p} = \frac{\gamma}{\gamma - 1} \frac{dT}{T}$$

c) From **a)** and **b)** calculate dT/dz, the change in temperature with increasing z.

d) If the pressure and temperature at sea-level are given by p_0 and T_0 respectively, and the atmosphere is considered to be adiabatic, find the pressure p at height z.

Problem 5

N identical classical particles occupy a square lattice with 2N sites, with at most one particle per site. Alternate sites are labeled A and B (see figure).



Denote by c the fraction of particles on the A sites, N_A, i.e.,

$$c = \frac{N_A}{N}$$

Since $N_A + N_B = N$ one obtains

$$\frac{N_B}{N} = \frac{N - N_A}{N} = 1 - c$$

Note that

$$0 \leq N_A, N_B \leq N$$

a) Compute the entropy for fixed c.

b) When two objects are on neighboring A and B sites, there is a repulsive interaction energy E_0. For fixed c, and assuming that all configurations at fixed c are equally likely, show that the average total energy of the system is

$$E(c) = 4NE_0c(1 - c)$$

c) In thermal equilibrium at temperature T, c is determined by minimizing the free energy

$$F(c) = E(c) - TS(c)$$

This system exhibits a second order phase transition at a temperature T_c.

(i) Describe the state of the system at very high temperatures $T \gg T_c$. What is the observed value of c?

(ii) Describe the state of the system at very low temperatures $T \ll T_c$. What are the possible values of c?

(iii) Determine T_c.

Mathematical Formulae

$$Z_c(T,V,1) = \int \frac{dq}{q_0} \exp[-\beta H]$$
$$Z_c(T,V,N) = \sum_n \exp[-\beta E_n]$$

$$F = -k_B T \ln Z_c$$
$$\Phi = -k_B T \ln Z_{gc}$$
$$F = U - TS$$
$$S = k_B \ln \Omega$$

Legendre Transformations from $f(x)$ **to** $g(p)$

$$g(p) = f(x) - px$$
$$p = \frac{df}{dx}$$

Integrals

$$\int_0^\infty dx \exp[-x^{\alpha_1}] = \Gamma\left(1 + \frac{1}{\alpha_1}\right)$$
$$\int \frac{dx}{x} = \ln x$$
$$\int x^2 dx = \frac{1}{3} x^3$$

Expansions

For $N \gg 1$

$$\ln N! = N \ln N - N$$

For $x \ll 1$

$$e^x = 1 + x + \ldots$$
$$\ln(1+x) = x - \frac{x^2}{2} + \ldots$$