

Quantum Mechanics Solutions

1. **(a)** If \mathcal{H} has an orthonormal basis consisting of the eigenvectors $\{|v_i\rangle\}$ of A with eigenvalues $\lambda_i \in \mathbb{C}$, then A can be written in terms of its spectral decomposition as $A = \sum_i \lambda_i |v_i\rangle\langle v_i|$. We are given in this case that $|\lambda_i| = 1, \forall i$. Notice that the adjoint of A is $A^\dagger = \sum_i \lambda_i^* |v_i\rangle\langle v_i|$. Then, we have

$$A^\dagger A = \sum_i \sum_j \lambda_i \lambda_j^* |v_j\rangle\langle v_j|v_i\rangle\langle v_i|.$$

Since $\{|v_i\rangle\}$ is orthonormal, $\langle v_j|v_i\rangle = \delta_{ij}$. Then, we have

$$A^\dagger A = \sum_i \sum_j \lambda_i \lambda_j^* \delta_{ij} |v_j\rangle\langle v_i| = \sum_i |\lambda_i|^2 |v_i\rangle\langle v_i| = \sum_i |v_i\rangle\langle v_i| = \hat{I} = AA^\dagger,$$

where we have used $|\lambda_i| = 1$ and the resolution of identity operator $\hat{I} = \sum_i |v_i\rangle\langle v_i|$ in terms of any orthonormal basis. Since $A^\dagger A = AA^\dagger = \hat{I}$, this proves that A is unitary.

(b) (i) Since we are asked to prove that two states are orthogonal to each other, it is natural to start with their inner product. Let $|\phi\rangle = T|\psi\rangle$, and let us start with $\langle\phi|\psi\rangle$:

$$\langle\phi|\psi\rangle = \langle T\phi|T\psi\rangle^* = \langle T\psi|T\phi\rangle = \langle T\psi|T^2\psi\rangle = \langle T\psi|-\psi\rangle = -\langle T\psi|\psi\rangle = -\langle\phi|\psi\rangle,$$

where we have used the definition of an anti-unitary operator (first step), an axiom of an inner product (second step), definition of $|\phi\rangle$ (third step), the given $T^2 = -I$ property (fourth step), linearity of an inner product in the second slot (fifth step), and again the definition of $|\phi\rangle$ (last step). Since we find $\langle\phi|\psi\rangle = -\langle\phi|\psi\rangle$, this means $\langle\phi|\psi\rangle = 0$ proving that $|\psi\rangle$ and $T|\psi\rangle$ are orthogonal to each other.

(ii) Let $|n\rangle$ be an eigenstate of H with eigenvalue E_n , $H|n\rangle = E_n|n\rangle$. We know E_n is real, since H is Hermitian. Starting with $[H, T] = \hat{O}$ (where \hat{O} is the zero operator) and applying this operator equality to the H eigenstate $|n\rangle$, we find

$$HT|n\rangle - TH|n\rangle = |0\rangle \Rightarrow H(T|n\rangle) = T(H|n\rangle) = T(E_n|n\rangle) = E_n(T|n\rangle).$$

Notice that in the second to last step above, we used the fact that E_n is real [$T(E_n|n\rangle)$ is actually $E_n^*T|n\rangle$ from the definition of an anti-unitary operator]. Since we see from above that $H(T|n\rangle) = E_n(T|n\rangle)$, the state $T|n\rangle$ has the same eigenvalue E_n as that of $|n\rangle$. This does not automatically imply degeneracy, since the state $T|n\rangle$ might differ from $|n\rangle$ by only a pure phase factor. However, this is not possible, since it was proven in part **(i)** that the states $T|n\rangle$ and $|n\rangle$ are orthogonal to each other. Hence, $T|n\rangle$ must be a distinct state compared to $|n\rangle$, which proves that H has a degenerate spectrum. If H were not Hermitian, this would not hold true in general, since the H eigenvalue E_n in that case could, in principle, be complex. $|n\rangle$ and $T|n\rangle$ would be different H eigenstates (with eigenvalues E_n and E_n^* , respectively).

2. (a) Writing down the representation of the A operator in the $\{|1\rangle, |2\rangle, |3\rangle\}$ orthonormal basis, it is straightforward to calculate the eigenvalues of A as a_0 (doubly degenerate eigenvalue with eigenvectors $|2\rangle$ and $\frac{1}{\sqrt{2}}(|1\rangle + |3\rangle)$), and $-a_0$ with eigenvector $\frac{1}{\sqrt{2}}(|1\rangle - |3\rangle)$. If a measurement of A on $|\psi\rangle$ is sure to yield a_0 , then $|\psi\rangle$ must lie in the degenerate subspace spanned by the two eigenvectors of A with eigenvalue a_0 (that is, $|\psi\rangle$ is, in general, a linear combination of the eigenvectors of A with eigenvalue a_0). Hence, $|\psi\rangle$ must be of the form:

$$|\psi\rangle = \alpha|1\rangle + \beta|2\rangle + \alpha|3\rangle,$$

for some $\alpha, \beta \in \mathbb{C}$. The second statement implies that b_0 is an eigenvalue of the B operator. Let $|\chi\rangle$ be the corresponding eigenvector, i.e. $B|\chi\rangle = b_0|\chi\rangle$. Expressing both the B operator and the $|\chi\rangle$ eigenvector in the $\{|1\rangle, |2\rangle, |3\rangle\}$ orthonormal basis and writing down the eigenvalue equation, one finds (after solving a linear equation) that $|\chi\rangle = \frac{1}{\sqrt{2}}(|1\rangle - i|2\rangle)$. Notice that we do not need to diagonalize the entire B matrix (which would be cumbersome); all we need is the eigenvector of B with eigenvalue b_0 . Since the probability of obtaining an eigenvalue of a Hermitian operator (when a measurement of that operator is made on some state $|\psi\rangle$) is given by the modulus square of the inner product of $|\psi\rangle$ with the corresponding eigenvector, the second statement means that $\langle\psi|\chi\rangle = 0$. This means that $\alpha + i\beta = 0$. Finally, we impose the normalization of $|\psi\rangle$ and find

$$|\psi\rangle = \frac{1}{\sqrt{3}}|1\rangle + \frac{i}{\sqrt{3}}|2\rangle + \frac{1}{\sqrt{3}}|3\rangle.$$

- (b) Let $|\phi_-\rangle$ denote $|\phi(t = 0^-)\rangle$. The normalized $|\phi_-\rangle$ is $\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$. The normalized eigenvectors of A with eigenvalue a_0 are $|\gamma_1\rangle = |2\rangle$ and $|\gamma_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |3\rangle)$. The reduction postulate tells us that if the state of the system just before the measurement of A is $|\phi_-\rangle$, then the state of the system right after the measurement will be

$$|\phi(t = 0^+)\rangle = \frac{\langle\gamma_1|\phi_-\rangle\langle\gamma_1| + \langle\gamma_2|\phi_-\rangle\langle\gamma_2|}{|\langle\gamma_1|\phi_-\rangle|^2 + |\langle\gamma_2|\phi_-\rangle|^2} = \frac{1}{\sqrt{6}}|1\rangle + \sqrt{\frac{2}{3}}|2\rangle + \frac{1}{\sqrt{6}}|3\rangle.$$

Now diagonalizing the Hamiltonian, it is straightforward to write down its eigenvectors and eigenvalues as $|h_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ ($\lambda_1 = \hbar\omega$); $|h_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$ ($\lambda_2 = -\hbar\omega$); $|h_3\rangle = |3\rangle$ ($\lambda_3 = 2\hbar\omega$). Expressing $|\phi(t = 0^+)\rangle$ in terms of the Hamiltonian eigenstates, we find $|\phi(t = 0^+)\rangle = \frac{\sqrt{3}}{2}|h_1\rangle - \frac{1}{2\sqrt{3}}|h_2\rangle + \frac{1}{\sqrt{6}}|h_3\rangle$. Using the time-evolution postulate, one then finds

$$|\phi(t)\rangle = \frac{\sqrt{3}}{2}|h_1\rangle e^{-i\omega t} - \frac{1}{2\sqrt{3}}|h_2\rangle e^{i\omega t} + \frac{1}{\sqrt{6}}|h_3\rangle e^{-2i\omega t}.$$

3. **(a) (i)** Since $[\Pi, H_0] = \hat{0}$, V_0 is an even function of x , and H_0 eigenstates $\psi_n^{(0)}$ are also parity eigenstates (even, odd, even, ... as $n = 0, 1, 2, \dots$) First-order correction to the ground state energy can be written as the expectation value of the perturbing potential in the ground state of H_0 . Using integration by parts, we have

$$E_0^{(1)} = \langle \psi_0 | H_1 | \psi_0 \rangle = \lambda \int_{-\infty}^{\infty} \psi_0 x \psi_0' \psi_0 dx = \lambda x \psi_0^3 \Big|_{-\infty}^{\infty} - \lambda \int_{-\infty}^{\infty} \psi_0^3 dx - 2\lambda \int_{-\infty}^{\infty} x \psi_0' \psi_0^2 dx.$$

The surface term vanishes as ψ_0 dies sufficiently fast. Then, we have

$$3 \int_{-\infty}^{\infty} x \psi_0' \psi_0^2 dx = - \int_{-\infty}^{\infty} \psi_0^3 dx \quad \Rightarrow \quad E_0^{(1)} = -\frac{c\lambda}{3}$$

- (ii)** The second-order correction to the first excited state can be written as the sum

$$E_1^{(2)} = \sum_{m \neq 1} \frac{|\langle \psi_m^{(0)} | H_1 | \psi_1^{(0)} \rangle|^2}{E_1^{(0)} - E_m^{(0)}} = \frac{|\langle \psi_0^{(0)} | H_1 | \psi_1^{(0)} \rangle|^2}{E_1^{(0)} - E_0^{(0)}} + \sum_{m \geq 2} \frac{|\langle \psi_m^{(0)} | H_1 | \psi_1^{(0)} \rangle|^2}{E_1^{(0)} - E_m^{(0)}}.$$

Now, since $\psi_0(x)$ is an even function of x , $\psi_0'(x)$ is odd, which makes $\psi_0^{(0)} x \psi_0'(x) \psi_1^{(0)}$ an odd function of x . Hence, the first ($m = 0$) term vanishes. For the remaining ($m \geq 2$) terms, the numerator is always positive and $E_1^{(0)} - E_m^{(0)} < 0$ which leads to $E_1^{(2)} < 0$.

(b) Since the given $\psi(x)$ (which is odd and has one node) is orthogonal to the exact ground state wavefunction $\phi_0(x) = \sqrt{\frac{1}{a}} \cos(\frac{\pi x}{2a})$ (which is even and nodeless), it can be used to get an estimate for the *first excited state* energy. By inspection, it also looks very similar to the true first excited state wavefunction. Normalizing $\psi(x)$, we get $\psi(x) = \sqrt{\frac{105}{16a^3}} x(x^2 - a^2)$. Using the variational principle for the 1st excited state (since $\langle \psi | \phi_0 \rangle = 0$) we have

$$E_1 < \langle H \rangle_{|\psi\rangle} = \int_{-a}^a dx \psi(x) \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \right] \psi(x) = \frac{21\hbar^2}{4ma^2} = 5.25 \frac{\hbar^2}{ma^2}.$$

This estimate is only 6.4% larger than the true first excited state energy of $E_1 = \frac{\hbar^2 \pi^2 (2)^2}{2m(2a)^2} \approx 4.93(\frac{\hbar^2}{ma^2})$.

4. (a) The parity of $|\phi_{n\ell m}\rangle$ eigenstates of the hydrogen atom is $(-1)^\ell$ (due to the parity of the spherical harmonics $Y_{\ell m}$). Since the uncertainty of the parity operator is finite, the given $|\psi\rangle$ cannot be a parity eigenstate, hence, it must be a linear combination of an even and odd eigenstate. Assume for the time being that $\ell_1 = \text{even}$ (with parity eigenvalue $+1$) and $\ell_2 = \text{odd}$ (with parity eigenvalue -1). Then we have (using $\Pi^2 = I$)

$$\langle \Delta\Pi \rangle_{|\psi\rangle}^2 = \langle \Pi^2 \rangle_{|\psi\rangle} - \langle \Pi \rangle_{|\psi\rangle}^2 = 1 - \left[|c_1|^2(+1) + (1 - |c_1|^2)(-1) \right]^2 = 1 - (2|c_1|^2 - 1)^2 = \frac{3}{4},$$

which implies that $|c_1|^2 = \frac{1}{4}$ or $\frac{3}{4}$ and $|c_2|^2 = 1 - |c_1|^2 = \frac{3}{4}$ or $\frac{1}{4}$. Then, $|\psi\rangle$ becomes (up to an overall phase)

$$|\psi\rangle = \frac{1}{2}|\phi_{n_1\ell_1 m_1}\rangle + \frac{\sqrt{3}e^{i\delta}}{2}|\phi_{n_2\ell_2 m_2}\rangle.$$

where δ is some arbitrary real number. Now, the H-atom energy eigenvalues are $-13.6 \text{ eV}/n^2$ ($n = 1, 2, \dots$). Notice that the Hamiltonian expectation value of -4.0375 eV , which is the average of two H-atom energy eigenvalues with weights of $|c_1|^2 = 1/4$ and $|c_2|^2 = 3/4$, falls between the $n = 1$ (-13.6 eV) and $n = 2$ (-3.4 eV) energy eigenvalues. Hence, one of the states must have the principal quantum number of $n = 1$, otherwise there would be no way of averaging two numbers which are both greater than or equal to -3.4 eV and come up with an average of -4.0375 eV . Choosing $n_1 = 1$ and looking for an (integer) solution for n_2 as

$$\langle H \rangle_{|\psi\rangle} = \frac{1}{4} \left(-13.6 \text{ eV} \right) + \frac{3}{4} \frac{(-13.6 \text{ eV})}{n_2^2} \stackrel{?}{=} -4.0375 \text{ eV},$$

we can obtain a solution with $n_2 = 4$. If n_2 were chosen as 1 (so that the weight of the -13.6 eV eigenvalue would be $3/4$), it can easily be shown that there is no integer (or real, for that matter) solution for n_1 that would satisfy the H expectation value requirement. Since $n_1 = 1$, this implies $\ell_1 = 0 = m_1$ as $0 \leq \ell_1 < n_1$ and $-\ell_1 \leq m_1 \leq \ell_1$. That $|\psi\rangle$ must be an eigenstate of the L_z operator (third piece of information in the problem) also constrains $m_2 = 0$. The value of ℓ_2 , on the other hand, cannot be determined uniquely. We know $0 \leq \ell_2 < 4$ and ℓ_2 has to be odd (due to the parity argument above). Hence, $\ell_2 = 1$ or 3 . Therefore,

$$|\psi\rangle = \frac{1}{2}|\phi_{100}\rangle + \frac{\sqrt{3}e^{i\delta}}{2}|\phi_{4\ell_2 0}\rangle, \quad \text{with } \ell_2 = 1 \text{ or } 3.$$

(b) Define the total spin operator $\vec{S} = \vec{S}_1 + \vec{S}_2$. We are free to choose the coordinate axis so that \vec{A} is aligned along the z -axis, so that $\vec{A} \cdot (\vec{S}_1 + \vec{S}_2) = |\vec{A}|(S_{1z} + S_{2z}) = AS_z$ where S_z is the z component of the total spin operator. Writing $\vec{S}_1 \cdot \vec{S}_2 = (\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2)/2$, we can express the given Hamiltonian as

$$H = \frac{A}{2\hbar^2}(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2) + \frac{A}{\hbar}S_z$$

The set of operators $\{\vec{S}_1^2, \vec{S}_2^2, \vec{S}^2, S_z\}$ commutes with H , hence the eigenstates can be labeled (dropping the S_1, S_2 indices for brevity) with kets of the form $|S, M\rangle$ where $-S \leq M \leq S$. Since $\vec{S}_i^2 = 2\hbar^2 I$ ($i = 1, 2$) for spin 1 objects, the above H can further be simplified in the $S_1 = S_2 = 1$ sector as

$$H = \frac{A}{2\hbar^2}(\vec{S}^2 - 4\hbar^2) + \frac{A}{\hbar}S_z$$

From angular momentum addition rules, with $S_1 = S_2 = 1$, $S = 0, 1$, or 2 (with $|M| \leq S$ for each S). Using $\vec{S}^2|S, M\rangle = S(S+1)\hbar^2|S, M\rangle$ and $S_z|S, M\rangle = M\hbar|S, M\rangle$, the eigenvalues of the Hamiltonian corresponding to the $|2, 2\rangle, |2, 1\rangle, |2, 0\rangle, |2, -1\rangle, |2, -2\rangle, |1, 1\rangle, |1, 0\rangle, |1, -1\rangle$, and $|0, 0\rangle$ states can be computed straightforwardly as $3A, 2A, A, 0, -A, 0, -A, -2A$, and $-2A$, respectively. Since $A > 0$, the ground state energy is $-2A$ (two-fold degenerate with eigenvectors $|1, -1\rangle$ and $|0, 0\rangle$). The first excited state energy is $-A$ (also two-fold degenerate with eigenvectors $|2, -2\rangle$ and $|1, 0\rangle$).

5. **(a)** For a single particle under a one-dimensional harmonic oscillator potential of frequency ω , the spectrum is $E_n = (n + \frac{1}{2})\hbar\omega$ ($n = 0, 1, 2, \dots$). Since the particles are non-interacting, the total energy of the 4-particle system can be written as a sum of one-particle energies. Since the particles are bosons, one can place any number of them in any single-particle state. The total energy of the system can then be determined by identifying how many particles are in each single-particle state. The energy eigenstates for the 4-particle system can be labeled by 4 integers (n_1, n_2, n_3, n_4) as $E_{n_1 n_2 n_3 n_4} = (n_1 + n_2 + n_3 + n_4)\hbar\omega + 2\hbar\omega$ ($n_i = 0, 1, 2, \dots$). Or, one can use a single quantum number $N = n_1 + n_2 + n_3 + n_4$ ($N = 0, 1, 2, \dots$) to label the energy eigenstates. Let $n = 0, 1, 2, \dots$ denote a generic single-particle state. Clearly, the ground state is obtained with $N = 0$ (placing all particles in $n = 0$), the first excited state is obtained with $N = 1$ (placing one of the particles in $n = 1$, the other three in $n = 0$). For the second excited state ($N = 2$), one can place two particles in the $n = 0$ and the other two in $n = 1$, *OR* three of them in $n = 0$ and one of them in $n = 2$ (hence, the second excited state is 2-fold degenerate). For the third excited state ($N = 3$), the energy is $E_3 = 3\hbar\omega + 2\hbar\omega = 5\hbar\omega$. This state can be obtained by **(A)** placing one particle in $n = 0$ and the other three in $n = 1$, *or* **(B)** placing two particles in $n = 0$, one particle in $n = 1$, and one particle in $n = 2$, *or* **(C)** placing three particles in the $n = 0$ and the remaining one in $n = 3$. Hence, the $N = 3$ third-excited state of the 4-particle system has an energy of $5\hbar\omega$ and is 3-fold degenerate.

To write the normalized wavefunctions for the 4-particle system, we use products of single-particle wavefunctions $\psi_n(x_i)$. Since the particles are identical bosons, we should not be able to distinguish which particle is in which particular single-particle state. That is, the 4-particle wavefunctions should be properly symmetrized. For the cases of **(A)** and **(C)** above, this means that there will be $4!/3! = 4$ product wavefunctions in the expansion (for example, in **A**, we should not be able to tell which particle is in the $n = 0$ state) that results in the following normalized wavefunctions

$$\begin{aligned}\psi_A(x_1, x_2, x_3, x_4) &= \frac{1}{2}(\psi_0(x_1)\psi_1(x_2)\psi_1(x_3)\psi_1(x_4) + \psi_1(x_1)\psi_0(x_2)\psi_1(x_3)\psi_1(x_4) + \\ &\quad \psi_1(x_1)\psi_1(x_2)\psi_0(x_3)\psi_1(x_4) + \psi_1(x_1)\psi_1(x_2)\psi_1(x_3)\psi_0(x_4)) \\ \psi_C(x_1, x_2, x_3, x_4) &= \frac{1}{2}(\psi_3(x_1)\psi_0(x_2)\psi_0(x_3)\psi_0(x_4) + \psi_0(x_1)\psi_3(x_2)\psi_0(x_3)\psi_0(x_4) \\ &\quad \psi_0(x_1)\psi_0(x_2)\psi_3(x_3)\psi_0(x_4) + \psi_0(x_1)\psi_0(x_2)\psi_0(x_3)\psi_3(x_4))\end{aligned}$$

For **(B)**, since two particles are in the same ($n = 0$) state, there will be a total of $4!/2! = 12$ product wavefunctions:

$$\begin{aligned}\psi_B(x_1, x_2, x_3, x_4) &= \frac{1}{\sqrt{12}}(\psi_0(x_1)\psi_0(x_2)\psi_1(x_3)\psi_2(x_4) + \psi_0(x_1)\psi_0(x_2)\psi_2(x_3)\psi_1(x_4) + \psi_1(x_1)\psi_0(x_2)\psi_0(x_3)\psi_2(x_4) + \\ &\quad \psi_2(x_1)\psi_0(x_2)\psi_0(x_3)\psi_1(x_4) + \psi_1(x_1)\psi_2(x_2)\psi_0(x_3)\psi_0(x_4) + \psi_2(x_1)\psi_1(x_2)\psi_0(x_3)\psi_0(x_4) + \\ &\quad \psi_0(x_1)\psi_1(x_2)\psi_0(x_3)\psi_2(x_4) + \psi_0(x_1)\psi_2(x_2)\psi_0(x_3)\psi_1(x_4) + \psi_0(x_1)\psi_1(x_2)\psi_2(x_3)\psi_0(x_4) + \\ &\quad \psi_0(x_1)\psi_2(x_2)\psi_1(x_3)\psi_0(x_4) + \psi_1(x_1)\psi_0(x_2)\psi_2(x_3)\psi_0(x_4) + \psi_2(x_1)\psi_0(x_2)\psi_1(x_3)\psi_0(x_4))\end{aligned}$$

(b) For a single particle under a two-dimensional harmonic oscillator potential of frequency ω , the spectrum is $E_{n_x, n_y} = (n_x + n_y + 1)\hbar\omega$ ($n_x = 0, 1, \dots$ and $n_y = 0, 1, \dots$). Adding also the spin degree of freedom, let us label the single-particle states with $|n_x, n_y, m_s\rangle$ (where $m_s = \uparrow$ or \downarrow). For non-interacting many particle systems, the total energy will simply be the sum of single-particle energies (one cannot, however, put more than 1 electron in a given single-particle quantum state). For $N = 1$ particle, the ground state will have a degeneracy, since $|0, 0, \uparrow\rangle$ and $|0, 0, \downarrow\rangle$ states have the same energy. For $N = 2$ particle system, on the other hand, one of the electrons can be in the $|0, 0, \uparrow\rangle$ state, while the other can be in the $|0, 0, \downarrow\rangle$. For this case, the ground state is non-degenerate. The next value of N for which the ground state will be non-degenerate will be when the electrons occupy all of the single-particle states that are in the next “shell”. These would be the single-particle states $|1, 0, \uparrow\rangle$, $|1, 0, \downarrow\rangle$, $|0, 1, \uparrow\rangle$, and $|0, 1, \downarrow\rangle$, which all have the same energy. If all these states are occupied, the corresponding ground state will again be non-degenerate. This requires 2 (to fill the $n_x = n_y = 0$ states) plus 4 (to fill $n_x + n_y = 1$ states) electrons, i.e. a total of $N = 6$ electrons. Therefore, the lowest two values of N for which the ground state is non-degenerate are $N = 2$ and 6.

With $N = 2$, the first excited state is obtained when an electron gets transferred from either the $|0, 0, \uparrow\rangle$ or $|0, 0, \downarrow\rangle$ to one of the 4 possible single-particle states mentioned above (with $n_x + n_y = 1$). There would be 1 electron with $n_x = n_y = 0$ (contributing an energy of $\hbar\omega$) and another with $n_x + n_y = 1$ (contributing an energy of $2\hbar\omega$). Hence, the total energy of the first excited state will be $E_1 = \hbar\omega + 2\hbar\omega = 3\hbar\omega$. Since the electron to be transferred from the $n_x = n_y = 0$ level can be chosen in 2 ways (up or down spin) and put into one of the 4 possible single-particle states (with $n_x + n_y = 1$ and up/down spin), the degeneracy of the first excited state of the $N = 2$ particle system will be $2 \times 4 = 8$ -fold.

With $N = 6$, the first excited state is obtained when an electron gets transferred from one of the 4 states (with $n_x + n_y = 1$ and up/down spin) to a single-particle state in the next “shell”. There are a total of 6 single-particle states in the next shell (with $n_x + n_y = 2$ and up/down spins). Altogether, in this $N = 6$ electron system, there are 2 electrons with $n_x = n_y = 0$ (each contributing an energy of $\hbar\omega$), 3 electrons with $n_x + n_y = 1$ (each contributing an energy of $2\hbar\omega$), and 1 electron with $n_x + n_y = 2$ contributing an energy of $3\hbar\omega$. Hence, the total energy of the first excited state will be $E_1 = 2\hbar\omega + 6\hbar\omega + 3\hbar\omega = 11\hbar\omega$. Since the electron to be transferred from the $n_x + n_y = 1$ level can be chosen in 4 ways and put into one of the 6 possible single-particle states (with $n_x + n_y = 2$ and up/down spin), the degeneracy of the first excited state of the $N = 6$ particle system will be $4 \times 6 = 24$ -fold.