Quantum Mechanics Problems

1. **(a)** Let A be a linear operator on a finite-dimensional Hilbert space $\mathcal{H}$. Assume that $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of A. Show that if all the eigenvalues of A have modulus 1, then A must be a unitary operator.

   (b) An operator on a Hilbert space $\mathcal{H}$ is called "anti-unitary" if it satisfies the following for all states $|\phi\rangle, |\chi\rangle \in \mathcal{H}$ and c_1, c_2 are complex numbers with complex conjugates $c_1^\star, c_2^\star$, respectively:

$$\begin{aligned} T(c_1|\phi\rangle + c_2|\chi\rangle) &= c_1^\star T|\phi\rangle + c_2^\star T|\chi\rangle \\ \langle\phi|\chi\rangle &= \langle T\phi|T\chi\rangle^\star. \end{aligned}$$

   If T is an anti-unitary operator such that $T^2 = -\hat{I}$ where $\hat{I}$ is the identity operator and $|\psi\rangle$ is an arbitrary state in $\mathcal{H}$, show that **(i)** $|\psi\rangle$ are $T|\psi\rangle$ are orthogonal to each other, and **(ii)** if a Hermitian operator H commutes with such a T operator, show that the spectrum of H must have some degeneracy. Would this hold true if H were not Hermitian?

2. Let A and B be two operators on a three-dimensional Hilbert space $\mathcal{H}$ with an orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$, and a_0 and b_0 are real numbers.

$$\begin{aligned} A &= a_0\Big(|1\rangle\langle 3| + |2\rangle\langle 2| + |3\rangle\langle 1|\Big) \\ B &= 2b_0\Big(|1\rangle\langle 1| + |2\rangle\langle 2|\Big) + b_0\Big(|3\rangle\langle 3| + |2\rangle\langle 3| + |3\rangle\langle 2|\Big) + ib_0\Big(|2\rangle\langle 1| + |3\rangle\langle 1| - |1\rangle\langle 2| - |1\rangle\langle 3|\Big) \end{aligned}$$

   (a) Construct the normalized state $|\psi\rangle \in \mathcal{H}$ consistent with both of the following statements:

   - If a measurement of A is performed on $|\psi\rangle$, the probability of obtaining the value a_0 is 100%.
   - If a measurement of B is made on $|\psi\rangle$, there is no chance of obtaining a value of b_0 (although a B measurement made on another state $|\chi\rangle \neq |\psi\rangle$ can possibly yield the value b_0).

   (b) Let $H = \hbar\omega(|1\rangle\langle 2| + |2\rangle\langle 1|) + 2\hbar\omega|3\rangle\langle 3|$ be the Hamiltonian for the system. Just before $t = 0$, let the system be in the state $|\phi(t = 0^-)\rangle = |1\rangle + |2\rangle$. A measurement of A is carried out on $|\phi\rangle$ at $t = 0$ and the value a_0 is obtained. Compute the normalized state of the system at a later time t.

3. **(a)** Consider a particle moving in a one-dimensional potential $V_0(x)$, which has a "well" structure so that there is at least one bound state. Assume that the unperturbed Hamiltonian $H_0 = p^2/2m + V_0(x)$ commutes with the parity operator. A perturbation $H_1 = \lambda x \psi_0'(x)$ is introduced, where $\psi_0(x)$ is the ground state wavefunction of H_0 and ψ_0' is its derivative with respect to x. Let $E_n^{(m)}$ denote the $m-$th order $(m = 1, 2, ...)$ correction to the $n-$th excited state $(n = 0, 1, ...)$ energy.
   (i) Compute $E_0^{(1)}$ and express your answer in terms of $c \equiv \int_{-\infty}^{\infty} dx\ \psi_0^3(x)$.
   (ii) Show that $E_1^{(2)} < 0$.

   (b) This problem is about estimating one of the energy eigenvalues for the particle-in-a-box problem (with infinite potential at the boundaries). Consider the function $\psi(x) = x(x^2 - a^2)$ defined in the interval $-a < x < a$ and is zero elsewhere. This "trial" wavefunction can be used to get a very good estimate of one of the stationary state energies of the particle-in-a-box problem. Explain which stationary state, obtain the corresponding estimate, and compare with the exact result.

4. **(a)** Let $|\phi_{n\ell m}\rangle$ denote the normalized Hamiltonian eigenstates for an electron moving in the Coulomb potential of a proton, where n, ℓ, and m denote the principal, angular momentum, and magnetic quantum numbers, respectively. Assume that the electron is in a superposition of two such eigenstates, $|\psi\rangle = c_1|\phi_{n_1\ell_1 m_1}\rangle + c_2|\phi_{n_2\ell_2 m_2}\rangle$. The following information is known about the state $|\psi\rangle$.

   - The uncertainty of the parity operator in the state $|\psi\rangle$ is $(\Delta\Pi)_\psi = \sqrt{3}/2$.
   - The Hamiltonian expectation value $\langle\psi|H|\psi\rangle = -4.0375$ eV.
   - $|\psi\rangle$ is an eigenstate of the $z-$component of the orbital angular momentum operator L_z.

   Based on this information, determine the values for $c_1, c_2, n_1, \ell_1, m_1, n_2, \ell_2, m_2$. If some of them cannot be determined uniquely from the given information, indicate the possible values that they can take.

   (b) Two spin 1 objects interact with each other and with an external vector field $\vec{A}$ in such a way that the Hamiltonian for the system can be written as

$$H = \frac{A}{\hbar^2}(\vec{S}_1 \cdot \vec{S}_2) + \frac{\vec{A} \cdot (\vec{S}_1 + \vec{S}_2)}{\hbar},$$

   where $A = |\vec{A}|$ is the magnitude of the field $\vec{A}$. Calculate the energies and degeneracies of the ground state and the first excited state of the system.

5. **(a)** Consider a system of 4 non-interacting identical $S = 0$ particles under a one-dimensional harmonic oscillator potential of angular frequency ω. Find the energy and the degeneracy of the third excited state of the system. If $\psi_n(x_i)$ denotes the normalized wavefunction for the i^{th} particle in the n^{th} excited state, write down all the normalized third excited state wavefunctions for this system.

   (b) Consider a system of N non-interacting electrons under a *two-dimensional* isotropic harmonic oscillator potential of angular frequency ω. Find the two lowest possible values of N such that the ground state of the system is non-degenerate. Then, for these values of N, find the first excited state energies and the corresponding degeneracies.