University of Illinois at Chicago
Department of Physics

Quantum Mechanics
Qualifying Examination

January 7, 2013 (Tuesday)
9:00 am - 12:00 noon

Full credit can be achieved from completely correct answers to **4 questions**. If the student attempts all 5 questions, all of the answers will be graded, and the **top 4 scores** will be counted toward the exam's total score.

Formulas

$$\int_0^\infty x^n e^{-ax} dx = n!/a^{n+1} \ , \quad \text{valid for complex } a \text{ as long as } \mathrm{Re}(a) > 0.$$

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} dx = \sqrt{\pi/\lambda} \ , \quad \text{valid for complex } a \text{ as long as } \mathrm{Re}(\lambda) \geq 0.$$

$$\int_{-\infty}^{\infty} x^2 e^{-\lambda x^2} dx = \frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}} \ , \quad \frac{\int_{-\infty}^{\infty} x^2 e^{-\lambda x^2} dx}{\int_{-\infty}^{\infty} e^{-\lambda x^2} dx} = \frac{1}{2\lambda} \ .$$

$$\text{Fourier transform: } \tilde{\psi}(k) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi(x) dx \ , \ \tilde{\psi}(x) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(k) e^{ikx} dk \ .$$

$$\langle \mathcal{O} \rangle = \int \int \int \Psi^*(\boldsymbol{x}) \mathcal{O} \Psi(\boldsymbol{x}) d^3x \ .$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

$$\sigma_x \sigma_y = i\sigma_z = -\sigma_y \sigma_x \ , \sigma_y \sigma_z = i\sigma_x = -\sigma_z \sigma_y \ , \sigma_z \sigma_x = i\sigma_y = -\sigma_x \sigma_z \ .$$

(1) **Gaussian Wave Packet**

A Gaussian wave packet describes the initial amplitude of a free non-relativistic one-dimensional quantum particle of mass $m = \frac{1}{2}$ (in units $\hbar = 1$),

$$\psi(x, t = 0) = N \exp(-3x^2 + 5x + i100x) \ .$$

(a) By completing the square of the **real** part of the exponent, determine the wave function normalization factor N.

(b) Find the mean position $\langle x \rangle$ and the mean wave number $\langle p \rangle$ of the particle.

(c) What is the wave-number amplitude $\tilde{\psi}(k)$ in the wave-number k-representation (or the momentum representation)?

(d) Find the uncertainties of the position and the momentum, $\sqrt{\langle (x - \langle x \rangle)^2 \rangle}$, $\sqrt{\langle (p - \langle p \rangle)^2 \rangle}$,

(e) What is the group velocity of the packet?

(f) How is this wave-number amplitude $\tilde{\psi}$ changed in time?

(g) Find $\psi(x, t)$. Your result can be in an integral form. Describe qualitatively how the wave function evolves.

(2) **Hydrogen bound states.**

In units $2m = 1, \hbar = 1$, the radial Schrödinger equation of the hydrogen atom is given by

$$\left[-\tfrac{d^2}{dr^2} - \tfrac{g^2}{r} + \tfrac{\ell(\ell+1)}{r^2} \right] u(r) = \varepsilon u(r) \ .$$

The lowest eigenstate of a given ℓ is known to have the form, $u^0_\ell(r) = C_\ell r^{\ell+1} \exp(-r/a_\ell)$.

(a) For a given ℓ, determine the eigenvalue ε^0_ℓ and the size parameter a_ℓ, in terms of the Coulomb strength g^2.

(b) The initial 3-dimensional wave function at $t = 0$ is the superposition of the above states $\ell = 0, 1$.

$$\psi(x, 0) = D \left(e^{-g^2 \frac{r}{2}} + g^2 r e^{-g^2 \frac{r}{4}} \cos\theta \right) \ .$$

Determine the quantum expectation average of $\langle \cos\theta \rangle$ as a function of time.

(3) **Planar Rotor and Perturbation.**

A permanent planar dipole $\boldsymbol{p}$, which lies on the x-y plane, is described by the rotation Hamiltonian $H_R = -\frac{\hbar^2}{2I} \frac{d^2}{d\phi^2}$, where ϕ is the angle of $\boldsymbol{p}$ with respect to the x axis.

(a) Write down the three lowest energy eigenvalues and their corresponding eigenstates. Arrange these states to be eigenstates of the angular momentum operator $L_z = -i\hbar \frac{d}{d\phi}$.

(b) A weak electric field $\boldsymbol{E}$ along the y axis is turned on. The interaction is given by $-\boldsymbol{p} \cdot \boldsymbol{E}$. Find all matrix elements of this perturbed energy operator between H_R eigenstates in (a). The result is expressed in terms of p,E,$\hbar$ and I.

(c) Determine the perturbed energies to the second order effect for the lowest lying states.

(4) **Fermi-Golden rule, scattering length, Born approximation.**
The asymptotic form of a scattering wave is given by $\psi(\boldsymbol{x}) \longrightarrow e^{i\mathbf{k}\cdot\mathbf{x}} + f(\theta)\frac{e^{ikr}}{r}$ for a particle of mass m in a spherical potential $V(r)$. The scattering amplitude is given by the Born approximation for a weak potential,

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'}V(r')d^3\mathbf{r}' \ .$$

(a) Carry out the angular integration in $f(\theta)$ so that only the radial integration remains. Simplify the result in terms of q ($\boldsymbol{q} = \boldsymbol{k'} - \boldsymbol{k}$).

(b) Find the differential cross section $d\sigma/d\Omega = |f(\theta)|^2$ for a weak delta-shell potential $V(r) = g\ \delta(r-R)$, located at the radius R.

(c) On the other hand, the cross-section can be derived from the Fermi Golden Rule about the transition rate from an initial state i to the final states f,

$$\Gamma = \tfrac{2\pi}{\hbar}|\langle f|H'|i\rangle|^2\rho(E_f) \ ,$$

where $\rho(E_f)$ counts the final state density when the system is confined in a very large cube with the periodic boundary condition. Derive the Born cross-section result from the Fermi golden rule by working out the state density, the solid angle differential, and the incident flux.

(d) The scattering problem can also be solved by the phase shift method. In the low energy limit of a very small k, the s-wave outside the potential range becomes a straight line $r\psi(r) = u(r) \longrightarrow\ A\times(r-a)$. Here A is an arbitrary multiplicative constant. The parameter a, i.e. the extrapolated intercept of the outside wave, is called the scattering length. As the s-wave effect dominates at the low k, we know that $f(\theta)\approx -a$.

Determine the scattering length a for the above delta-shell potential, in terms of the shell radius R and the strength g, by solving the corresponding radial Schrödinger equation at $k\approx 0$,

$$-\tfrac{d^2}{dr^2}u(r) + g\delta(r-R)u(r) = 0 \ .$$

Confront your result with Born approximation.

(5) **Coupled Angular Momenta.**
We study the composite system of two localized spin-half particles, 1 and 2. Their corresponding Pauli matrices are $\sigma_i^{(1)}$ and $\sigma_i^{(2)}$. The spin-spin interaction among them is described by

$$H = \sigma_x^{(1)}\sigma_x^{(2)} + \sigma_y^{(1)}\sigma_y^{(2)} + \sigma_z^{(1)}\sigma_z^{(2)} \ .$$

(a) Find the energy eigenstates and eigenvalues of H, by using the property of the angular momentum sum.

(b) Then find the energy eigenstates and eigenvalues of another Hamiltonian,

$$H^{(+)} = \sigma_y^{(1)}\sigma_y^{(2)} + \sigma_x^{(1)}\sigma_x^{(2)} \ .$$

(c) When the spin state $|\psi\rangle$ of *one* single spin-half particle is rotated about the y-axis by an angle β, the new state $|\psi'\rangle = U|\psi\rangle$ is described by the unitary transformation $U = \exp(-i\sigma_y\beta/2)$.

Work out the explicit entries in the matrix U in terms of β. The Pauli matrices σ_i $(i = x, y, z)$ when transformed become $\sigma'_i = U\sigma_i U^\dagger$. Work out the explicit relation that $\sigma'_x = c_1\sigma_x + c_2\sigma_z$ and express the coefficients c_1, c_2 in terms of the angle β. Do the same calculation for σ'_z and σ'_y. Explain the physical meaning of the transformation.

Show the result for the special case of $\beta = \pi$.

(d) Finally, If the relative sign of terms in $H^{(+)}$ is flipped to give the third Hamiltonian,

$$H^{(-)} = \sigma_y^{(1)}\sigma_y^{(2)} - \sigma_x^{(1)}\sigma_x^{(2)} .$$

How is $H^{(-)}$ related to $H^{(+)}$ by a unitary transformation? What is the energy eigenstates and eigenvalues of $H^{(-)}$?