Problem 1

Consider a disc of charge density $\sigma(\vec{r}) = \sigma_0|\vec{r}|$ and radius R that lies within the xy-plane. The origin of the coordinate systems is located at the center of the ring.

a) Give the potential at the point $\vec{P} = (0,0,z)$ in terms of $\sigma_0, R,$ and z.

b) We next put a conducting plane into the $z = d$ plane. The potential of the conducting plane is fixed at $V = 0$. Compute the total potential at a point $\vec{P} = (0,0,z)$.

c) If the total charge, Q, on the disc is fixed, find the charge density in terms of Q and use it to obtain the form of $\phi_{tot}\left(\vec{P}\right)$ in terms of Q, R, z in the limit $R \gg z, d$ up to leading order in (z/R)?

d) Give an explicit form of the induced charge density at $\vec{P} = (0,0,d)$ in the limit $R \gg d$ using the results of part c).

Solution:

a) The coordinate of a point on the disc is given by

$$\vec{r} = r(\cos\varphi, \sin\varphi, 0)$$

The potential at $\vec{P}$ is then given by

$$\begin{aligned}\phi\left(\vec{P}\right) &= \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{\left|\vec{P}-\vec{r}\right|} = \frac{1}{4\pi\varepsilon_0}\int d^2r \frac{\sigma_0|\vec{r}|}{\left|\vec{P}-\vec{r}\right|} = \frac{\sigma_0}{4\pi\varepsilon_0}\int_0^{2\pi} d\varphi \int_0^R r dr \frac{r}{\sqrt{z^2+(r\cos\varphi)^2+(r\sin\varphi)^2}} \\ &= \frac{\sigma_0}{4\pi\varepsilon_0}\int_0^{2\pi} d\varphi \int_0^R dr \frac{r^2}{\sqrt{z^2+r^2}} = \frac{\sigma_0}{2\varepsilon_0}\int_0^R dr \frac{r^2}{\sqrt{z^2+r^2}} \\ &= \frac{\sigma_0}{2\varepsilon_0}\left\{\frac{1}{2}R\sqrt{z^2+R^2} - \frac{1}{2}z^2 \ln\left[\frac{R+\sqrt{z^2+R^2}}{z}\right]\right\}\end{aligned}$$

where I used

$$\int dr \frac{r^2}{\sqrt{z^2+r^2}} = \frac{1}{2}r\sqrt{z^2+r^2} - \frac{1}{2}z^2 \ln\left[r+\sqrt{z^2+r^2}\right]$$

b) Using the method of images, the image charge is a disc of charge density $\rho(\vec{r}) = -\rho_0|\vec{r}|$ and radius R that is parallel to the conducting plane at a distance of $2d$ from the first disc. The resulting total potential is then

$$\phi_{tot}\left(\vec{P}\right) = \frac{\sigma_0}{2\varepsilon_0}\left\{\frac{1}{2}R\sqrt{z^2+R^2} - \frac{1}{2}z^2\ln\left[\frac{R+\sqrt{z^2+R^2}}{z}\right]\right\}$$
$$-\frac{\sigma_0}{2\varepsilon_0}\left\{\frac{1}{2}R\sqrt{(2d-z)^2+R^2} - \frac{1}{2}(2d-z)^2\ln\left[\frac{R+\sqrt{(2d-z)^2+R^2}}{(2d-z)}\right]\right\}$$

c) If the total charge, Q, on the disc is fixed, what is the form of $\phi_{tot}\left(\vec{P}\right)$ in terms of Q, R, z in the limit $R \gg z, d$ up to leading order in (z/R)?

If the charge is fixed, we have

$$Q = \int dq = \int_0^{2\pi} d\varphi \int_0^R r dr \sigma_0 r = 2\pi\sigma_0 \frac{R^3}{3}$$

or

$$\sigma_0 = \frac{3Q}{2\pi R^3}$$

and thus

$$\phi_{tot}\left(\vec{P}\right) = \frac{1}{2\varepsilon_0}\frac{3Q}{2\pi R^3}\left\{\frac{1}{2}R\sqrt{z^2+R^2} - \frac{1}{2}z^2\ln\left[\frac{R+\sqrt{z^2+R^2}}{z}\right]\right\}$$
$$-\frac{1}{2\varepsilon_0}\frac{3Q}{2\pi R^3}\left\{\frac{1}{2}R\sqrt{(2d-z)^2+R^2} - \frac{1}{2}(2d-z)^2\ln\left[\frac{R+\sqrt{(2d-z)^2+R^2}}{(2d-z)}\right]\right\}$$
$$= \frac{3Q}{8\pi\varepsilon_0 R^3}\left\{R\sqrt{z^2+R^2} - z^2\ln\left[\frac{R+\sqrt{z^2+R^2}}{z}\right]\right\}$$
$$-\frac{3Q}{8\pi\varepsilon_0 R^3}\left\{R\sqrt{(2d-z)^2+R^2} - (2d-z)^2\ln\left[\frac{R+\sqrt{(2d-z)^2+R^2}}{(2d-z)}\right]\right\}$$

We first consider the first two terms which yield

$$\frac{3Q}{8\pi\varepsilon_0 R}\left[\sqrt{1+\left(\frac{z}{R}\right)^2} - \sqrt{1+\left(\frac{2d-z}{R}\right)^2}\right]$$
$$\approx \frac{3Q}{16\pi\varepsilon_0 R^3}\left[z^2-(z-2d)^2\right] = \frac{3Q}{16\pi\varepsilon_0 R^3}\left[4zd-4d^2\right] = \frac{3Qd}{4\pi\varepsilon_0 R^3}[z-d]$$

The second set yields

$$- \frac{3Q}{8\pi\varepsilon_0 R^3}\left\{z^2 \ln\left[\frac{R + \sqrt{z^2 + R^2}}{z}\right] - (2d - z)^2 \ln\left[\frac{R + \sqrt{(2d - z)^2 + R^2}}{(2d - z)}\right]\right\}$$

$$\approx -\frac{3Q}{8\pi\varepsilon_0 R^3}\left\{z^2 \ln\left[\frac{2R}{z}\right] - (2d - z)^2 \ln\left[\frac{2R}{(2d - z)}\right]\right\}$$

This is the leading contribution.

d) Give an explicit form of the induced charge density at $\vec{P} = (0, 0, d)$ in the limit $R \gg d$ using the results of part c).

The charge density follows from

$$\sigma = -\varepsilon_0 \frac{\partial \phi_{tot}}{\partial(-z)}\bigg|_{z=d}$$

and thus

$$\sigma = \varepsilon_0 \frac{\partial \phi_{tot}}{\partial z}\bigg|_{z=d} = \frac{3Q}{8\pi R^3}\frac{\partial}{\partial z}\left\{z^2 \ln\left[\frac{z}{2R}\right] - (2d - z)^2 \ln\left[\frac{2d - z}{2R}\right]\right\}\bigg|_{z=d}$$

$$= \frac{3Q}{8\pi R^3}\left\{2z \ln\left[\frac{z}{2R}\right] + z^2 \frac{1}{z} + 2(2d - z)\ln\left[\frac{2d - z}{2R}\right] - (2d - z)^2 \frac{-1}{2d - z}\right\}\bigg|_{z=d}$$

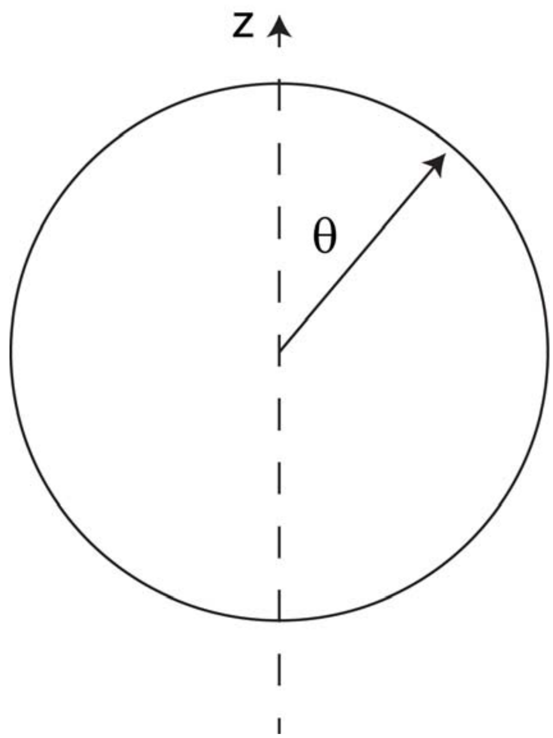
$$= \frac{3Q}{8\pi R^3}\left\{2z \ln\left[\frac{d}{2R}\right] + d + 2(2d - d)\ln\left[\frac{2d - d}{2R}\right] + 2d - d\right\}$$

$$= \frac{6Qd}{4\pi R^3}\left\{1 + \ln\left[\frac{d}{2R}\right]\right\}$$

Problem 2

Consider a sphere of radius R. The potential on the surface of the sphere varies as (see figure below) (free region inside and outside)

$$\phi(\theta) = \phi_0 \cos^2\theta$$



a) Compute the potential inside and outside of the sphere.

b) Compute the electric field inside the sphere.

c) Using Gauss; law, show that while the electric field inside the sphere is non-zero, no charge is contained inside the sphere.

Solution:

a) Since the problem has an aziumuthal symmetry, we have

$$\phi(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l(\cos\theta)$$

Using that

$$P_0(\cos\theta) = 1$$
$$P_2(\cos\theta) = \frac{1}{2}[3\cos^2\theta - 1]$$

we obtain

$$\phi(\theta) = \phi_0\cos^2\theta = \frac{\phi_0}{3}[2P_2(\cos\theta) + 1] = \frac{\phi_0}{3}[2P_2(\cos\theta) + P_0(\cos\theta)]$$

And thus

$$\begin{aligned}\int_{-1}^{1} dx\,\phi(R,x)P_m(x) &= \int_{-1}^{1} dx\,\frac{\phi_0}{3}[2P_2(\cos\theta) + P_0(\cos\theta)]P_m(x)\\ &= \frac{\phi_0}{3}\left[2\frac{2}{2m+1}\delta_{m,2} + \frac{2}{2m+1}\delta_{m,0}\right]\\ &= \frac{\phi_0}{3}\left[\frac{4}{5}\delta_{m,2} + 2\delta_{m,0}\right]\end{aligned}$$

If we want to evaluate the potential inside of the sphere, we need to set $B_l = 0$ and obtain

$$\begin{aligned}\int_{-1}^{1} dx\,\phi(R,x)P_m(x) &= \int_{-1}^{1} dx\left[\sum_l A_l R^l P_l(x)\right]P_m(x)\\ &= A_m R^m \frac{2}{2m+1}\end{aligned}$$

and thus for $m = 2$

$$\frac{\phi_0}{3}\frac{4}{5} = A_2 R^2 \frac{2}{5}$$
$$A_2 = \frac{2\phi_0}{3R^2}$$

and for $m = 0$

$$\frac{2\phi_0}{3} = 2A_0$$
$$A_0 = \frac{\phi_0}{3}$$

and thus

$$\phi(r,\theta) = \frac{\phi_0}{3}P_0(\cos\theta) + \frac{2\phi_0}{3}\left(\frac{r}{R}\right)^2 P_2(\cos\theta)$$

For the potential outside of the sphere, we set $A_l = 0$ and obtain

$$\int_{-1}^{1} dx\,\phi(R,x)P_m(x) = \int_{-1}^{1} dx\left[\sum_{l=0}^{\infty} B_l R^{-(l+1)} P_l(\cos\theta)\right]P_m(x)$$
$$= B_m R^{-(m+1)}\frac{2}{2m+1}$$

and thus for $m = 2$

$$\frac{\phi_0}{3}\frac{4}{5} = B_2 R^{-3}\frac{2}{5}$$
$$B_2 = \frac{2\phi_0}{3}R^3$$

and for $m = 0$

$$\frac{2\phi_0}{3} = 2\frac{B_0}{R}$$
$$B_0 = \frac{\phi_0}{3}R$$

and thus

$$\phi(r,\theta) = \frac{\phi_0}{3}\frac{R}{r}P_0(\cos\theta) + \frac{2\phi_0}{3}\left(\frac{R}{r}\right)^3 P_2(\cos\theta)$$

b) The electric field inside the sphere is then given by

$$\vec{E} = -\nabla\phi(r,\theta) = -\hat{r}\frac{\partial\phi(r,\theta)}{\partial r} - \hat{\theta}\frac{1}{r}\frac{\partial\phi(r,\theta)}{\partial\theta}$$
$$= -\hat{r}\frac{4\phi_0}{3}\frac{r}{R^2}P_2(\cos\theta) - \hat{\theta}\frac{1}{r}\frac{2\phi_0}{3}\left(\frac{r}{R}\right)^2[-3\sin\theta\cos\theta]$$
$$= -\frac{4\phi_0}{3}\frac{r}{R^2}P_2(\cos\theta)\hat{r} + 2\phi_0\frac{r}{R^2}[\sin\theta\cos\theta]\hat{\theta}$$

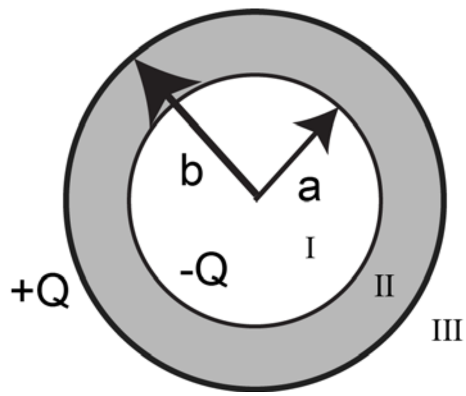
c) Using Gauss' law inside the shere

$$\oint \vec{E}\cdot d\vec{A} = -\frac{4\phi_0}{3}\frac{r}{R^2}r^2\int d\varphi\int_{-1}^{1} d(\cos\theta)P_2(\cos\theta) = 0$$

Thus, no charges are contained inside the sphere.

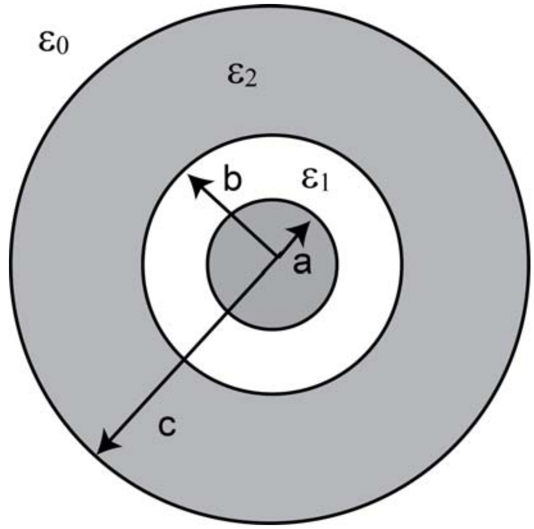
Problem 3

a) Consider the two conducting spheres with radii a and b ($b > a$) as shown in the figure below. The volume between the two spheres (region II) is filled with a material of permitivity ε. The permitivity in regions I and III is that of free space, ε_0. The two spheres are uniformly charged with total charge $\pm Q$.



(i) Compute the magnitude and direction of the electric field in regions I, II, and III.
(ii) Compute the capacitance of the two spheres.

b) Consider next two infinitely long concentric cylinders, as shown in the figure below. The inner cylinder of radius a is a conductor with linear charge density $\lambda_1 > 0$. The second cylinder with inner radius b and outer radius c consists of a material with permitivity ε_2 and is uniformly charged with line charge density $\lambda_2 < 0$ ($\lambda_1 > |\lambda_2|$). The space between the two cylinders (i.e., $a < r < b$) is filled with a medium of permitivity ε_1. The medium outside the outer cylinder possesses the permitivity ε_0. Compute the potential difference between a point at $|\vec{r}| = 2c$ (measured from the center of the inner cylinders) and the center of the inner cylinder.



Solutions:

a)

(i) Using Gauss' law, we can compute the electric fields of the two sphere system.

Region I:

The electric field in reggion I for $r < a,$ with r measured from the center of the sphere, is given by

$$\int \vec{E} \cdot d\vec{s} = E4\pi r^2 = \frac{q_{encl}}{\varepsilon_0} = 0$$
$$\Rightarrow E = 0$$

the electric field is zero inside the inner sphere.

Region II:
The electric displacement field for $a < r < b,$ with r measured from the center of the sphere is given by

$$\int D \cdot d\vec{s} = D4\pi r^2 = -Q$$
$$\Rightarrow \vec{D} = -\frac{Q}{4\pi r^2}\hat{r}$$
$$\Rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon} = -\frac{Q}{4\pi\varepsilon r^2}\hat{r}$$

The electric field points radially inward.

Region III:

The electric field in reggion III for $r > b,$ with r measured from the center of the sphere, is given by

$$\int \vec{E} \cdot d\vec{s} = E4\pi r^2 = \frac{q_{encl}}{\varepsilon_0} = 0$$
$$\Rightarrow E = 0$$

the electric field is zero inside the outside the outer sphere.

(ii) The potential difference between the two sphere is

$$\Delta\phi = -\int \vec{E}_{tot} \cdot d\vec{r} = \int_a^b \frac{Q}{4\pi\varepsilon} \frac{1}{r^2} dr$$
$$= \frac{Q}{4\pi\varepsilon} \left[\frac{1}{a} - \frac{1}{b} \right]$$

and the capacitance is thus

$$C = \frac{Q}{|\Delta\phi|} = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

b)We need to compute the electric field in the different regions of the problem.

i) $r < a$. We have $E = 0$ inside the inner cylinder is zero, since it is a conductor.
ii) $a < r < b$. We use

$$\int \vec{D} \cdot d\vec{a} = \lambda_1 l \Rightarrow 2\pi r l D = \lambda_1 l \Rightarrow D = \frac{1}{2\pi} \frac{\lambda_1}{r}$$

and thus

$$E = \frac{D}{\varepsilon_2} = \frac{1}{2\pi\varepsilon_2} \frac{\lambda_1}{r}$$

Note that since $\lambda_1 > 0$ the electric field points radially outwards.
iii) $b < r < c$. Note that the material in this region is insulating and uniformly charged. The volume charge density is

$$\rho = \frac{\lambda_3}{\pi(c^2 - b^2)}$$

and thus

$$\int \vec{D} \cdot d\vec{a} = \left(\lambda_1 + \lambda_3 \frac{(r^2 - b^2)}{(c^2 - b^2)} \right) l \Rightarrow 2\pi r l D = \left(\lambda_1 + \lambda_3 \frac{(r^2 - b^2)}{(c^2 - b^2)} \right) l$$
$$\Rightarrow D = \frac{1}{2\pi} \frac{1}{r} \left(\lambda_1 + \lambda_3 \frac{(r^2 - b^2)}{(c^2 - b^2)} \right)$$

and thus

$$E = \frac{D}{\varepsilon_3} = \frac{1}{2\pi\varepsilon_3} \frac{1}{r} \left(\lambda_1 + \lambda_3 \frac{(r^2 - b^2)}{(c^2 - b^2)} \right)$$

iv) $c < r$. Here we have

$$E = \frac{1}{2\pi\varepsilon_0} \frac{\lambda_1 + \lambda_3}{r}$$

We can now compute the potential difference

$$\Delta\Phi = -\int_{2c}^{0} \vec{E}_{tot} \cdot d\vec{r} = -\int_{2c}^{c} E\,dr - \int_{c}^{b} E\,dr - \int_{b}^{a} E\,dr - \int_{a}^{0} E\,dr$$

where

$$-\int_{2c}^{c} E\,dr = -\int_{2c}^{c} \frac{1}{2\pi\varepsilon_0}\frac{\lambda_1 + \lambda_3}{r}dr = \frac{\lambda_1 + \lambda_3}{2\pi\varepsilon_0}\ln 2$$

and

$$\begin{aligned}
-\int_{c}^{b} E\,dr &= -\int_{c}^{b} \frac{1}{2\pi\varepsilon_3}\frac{1}{r}\left(\lambda_1 + \lambda_3\frac{(r^2 - b^2)}{(c^2 - b^2)}\right)dr \\
&= -\int_{c}^{b} \frac{1}{2\pi\varepsilon_3}\frac{1}{r}\left(\lambda_1 - \lambda_3\frac{b^2}{(c^2 - b^2)}\right) - \int_{c}^{b} \frac{\lambda_3}{2\pi\varepsilon_3}\frac{r}{(c^2 - b^2)}dr \\
&= \frac{1}{2\pi\varepsilon_3}\left(\lambda_1 - \lambda_3\frac{b^2}{(c^2 - b^2)}\right)\ln\frac{c}{b} + \frac{\lambda_3}{4\pi\varepsilon_3}
\end{aligned}$$

and

$$-\int_{b}^{a} E\,dr = -\int_{b}^{a} \frac{1}{2\pi\varepsilon_2}\frac{\lambda_1}{r}dr = \frac{\lambda_1}{2\pi\varepsilon_2}\ln\frac{b}{a}$$

and

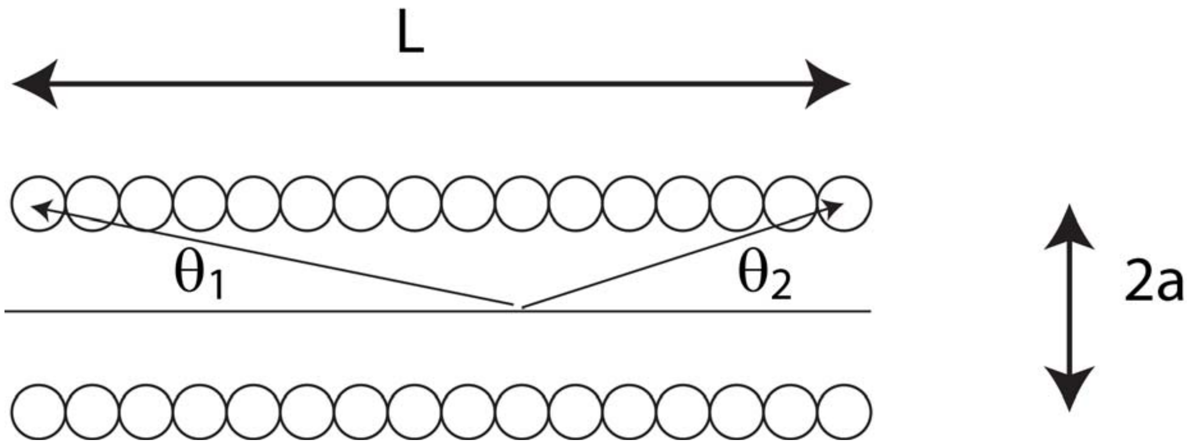
$$-\int_{a}^{0} E\,dr = 0$$

Thus I obtain

$$\Delta\Phi = \frac{\lambda_1 + \lambda_3}{2\pi\varepsilon_0}\ln 2 + \frac{1}{2\pi\varepsilon_3}\left[\left(\lambda_1 - \lambda_3\frac{b^2}{(c^2 - b^2)}\right)\ln\frac{c}{b} + \frac{\lambda_3}{2}\right] + \frac{\lambda_1}{2\pi\varepsilon_2}\ln\frac{b}{a}$$

Problem 4

A solenoid of finite length *L* and a readius *a* has *N* turns per unit length and carries a current *I*, with cicular cross section as shown in the figure below.



a) Compute the magnetic induction on the solenoid axis in the limit $NL \to \infty$ in terms of the angles θ_1 and θ_2.

b) For $a \gg L$, how does the magnetic induction scale with a?

Solutions:

a) We first compute the magnetic induction due to a single loop. Using Biot-Savart

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^3}$$

where

$$r = \sqrt{a^2 + z^2}$$

By symmetry, integration over the loop yields a magnetic induction along the z-axis. We then have

$$\begin{aligned} \vec{B} &= \frac{\mu_0}{4\pi} I \oint \frac{d\vec{l} \times \vec{r}}{r^3} \\ &= \frac{\mu_0}{4\pi} I \frac{1}{[a^2 + z^2]^{3/2}} \oint dl\, r \sin\varphi\, \hat{z} \end{aligned}$$

where

$$\sin\varphi = \frac{a}{[a^2 + z^2]^{1/2}}$$

and thus

$$\vec{B} = \frac{\mu_0}{4\pi} I \frac{1}{[a^2+z^2]^{3/2}} [a^2+z^2]^{1/2} \frac{a}{[a^2+z^2]^{1/2}} 2\pi a \hat{z}$$
$$= \frac{\mu_0}{2} I \frac{a^2}{[a^2+z^2]^{3/2}} \hat{z}$$

Next, we consider the entire solenoid.

$$\vec{B} = \hat{z}\frac{\mu_0}{2} NIa^2 \left[\int_0^{\frac{L}{2}+z} \frac{dz}{[a^2+z^2]^{3/2}} + \int_0^{\frac{L}{2}-z} \frac{dz}{[a^2+z^2]^{3/2}} \right]$$
$$= \hat{z}\frac{\mu_0}{2} NIa^2 \frac{1}{a^2} \left[\frac{\frac{L}{2}+z}{\left[a^2+\left(\frac{L}{2}+z\right)^2\right]^{1/2}} + \frac{\frac{L}{2}-z}{\left[a^2+\left(\frac{L}{2}-z\right)^2\right]^{1/2}} \right]$$
$$= \hat{z}\frac{\mu_0 NI}{2} [\cos\theta_1 + \cos\theta_2]$$

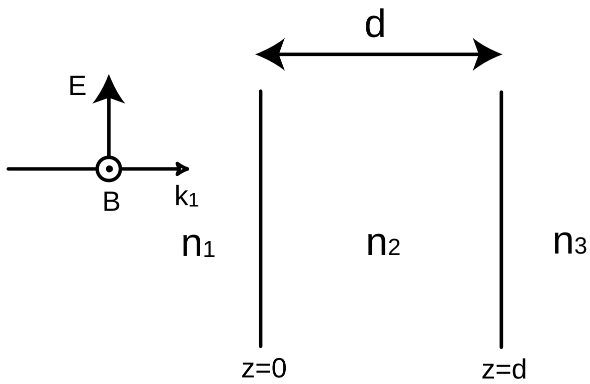
The limit $NL \to \infty$ is necessary to perform the integration over z.

b) From the above expression, we have

$$\lim_{a \gg L} \vec{B} = \lim_{a \gg L} \hat{z}\frac{\mu_0}{2} NI \left[\frac{\frac{L}{2}+z}{\left[a^2+\left(\frac{L}{2}+z\right)^2\right]^{1/2}} + \frac{\frac{L}{2}-z}{\left[a^2+\left(\frac{L}{2}-z\right)^2\right]^{1/2}} \right]$$
$$\approx \hat{z}\frac{\mu_0}{2} NI \left[\frac{\frac{L}{2}+z}{a} + \frac{\frac{L}{2}-z}{a} \right] = \hat{z}\frac{\mu_0 I}{2a} NL$$

Problem 5

An electromagnetic plane wave is incident perpendicular to a layered interface, as shown in the figure below. The indices of refraction of the three media is $n_1, n_2 = 2n$ and $n_3 = 4n_1$ while the permeability of all three regions is the same, μ_0. The thickness of the intermediate layer is d. Each of the other media is semi-infinite.



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a) State the boundary conditions at both interfaces in terms of the electric fields.

b) Compute the ratio between the incident electric field in medium 1 and the transmitted electric field in medium 3, i..e, compute $|E_i/E_t|^2$.

c) If the thickness d is varied, the ratio $|E_i/E_t|^2$ oscillates. What is the period of the oscillation? For which values of d is $|E_i/E_t|^2$ the smallest?

Solutions:

a) State the boundary conditions at both interfaces in terms of the electric fields.

The EM wave contains only components that are perpendicular to the interface. In region 1, there is an incoming and a reflected wave, in region 2 there is a right-moving and a left-moving wave, and in region 3, there is only a transmitted wave. Thus the boundary conditions at $z = 0$ are

$$E_i + E_r = E_+ + E_-$$
$$\frac{E_i - E_r}{c_1} = \frac{E_+ - E_-}{c_2} \Rightarrow E_i - E_r = \frac{n_2}{n_1}(E_+ - E_-) = 2(E_+ - E_-)$$

and at $z = d$ we have

$$E_+ e^{ik_2 d} + E_- e^{-ik_2 d} = E_t e^{ik_3 d}$$
$$\frac{E_+ e^{ik_2 d} - E_- e^{-ik_2 d}}{c_2} = \frac{E_t e^{ik_3 d}}{c_3} \Rightarrow E_+ e^{ik_2 d} - E_- e^{-ik_2 d} = \frac{n_3}{n_2} E_t e^{ik_3 d} = 2E_t e^{ik_3 d}$$

b) Compute the ratio between the incident electric field in medium 1 and the transmitted electric field in medium 3, i..e, compute $|E_i/E_t|^2$.

From the last two equations, I obtain

$$E_+ = \frac{1}{2}\left(1 + \frac{n_3}{n_2}\right) E_t e^{ik_3 d} e^{-ik_2 d} = \frac{3}{2} E_t e^{ik_3 d} e^{-ik_2 d}$$
$$E_- = \frac{1}{2}\left(1 - \frac{n_3}{n_2}\right) E_t e^{ik_3 d} e^{ik_2 d} = -\frac{1}{2} E_t e^{ik_3 d} e^{ik_2 d}$$

and from the first two equations

$$
\begin{aligned}
2E_i &= \left(1 + \frac{n_2}{n_1}\right)E_+ + \left(1 - \frac{n_2}{n_1}\right)E_- = 3E_+ - E_- \\
&= \left(1 + \frac{n_2}{n_1}\right)\frac{1}{2}\left(1 + \frac{n_3}{n_2}\right)E_t e^{ik_3 d} e^{-ik_2 d} + \left(1 - \frac{n_2}{n_1}\right)\frac{1}{2}\left(1 - \frac{n_3}{n_2}\right)E_t e^{ik_3 d} e^{ik_2 d} \\
&= \frac{9}{2}E_t e^{ik_3 d} e^{-ik_2 d} + \frac{1}{2}E_t e^{ik_3 d} e^{ik_2 d}
\end{aligned}
$$

and thus

$$
\begin{aligned}
4\frac{E_i}{E_t} &= e^{ik_3 d} e^{-ik_2 d}\left[\left(1 + \frac{n_2}{n_1}\right)\left(1 + \frac{n_3}{n_2}\right) + \left(1 - \frac{n_2}{n_1}\right)\left(1 - \frac{n_3}{n_2}\right)e^{i2k_2 d}\right] \\
&= e^{ik_3 d} e^{-ik_2 d}\left[9 + e^{i2k_2 d}\right]
\end{aligned}
$$

and hence

$$
\begin{aligned}
16\left|\frac{E_i}{E_t}\right|^2 &= \left[\left(1 + \frac{n_2}{n_1}\right)\left(1 + \frac{n_3}{n_2}\right) + \left(1 - \frac{n_2}{n_1}\right)\left(1 - \frac{n_3}{n_2}\right)e^{i2k_2 d}\right] \\
&\quad \times \left[\left(1 + \frac{n_2}{n_1}\right)\left(1 + \frac{n_3}{n_2}\right) + \left(1 - \frac{n_2}{n_1}\right)\left(1 - \frac{n_3}{n_2}\right)e^{-i2k_2 d}\right] \\
&= \left(1 + \frac{n_2}{n_1}\right)^2\left(1 + \frac{n_3}{n_2}\right)^2 + \left(1 - \frac{n_2}{n_1}\right)^2\left(1 - \frac{n_3}{n_2}\right)^2 \\
&\quad + \left(1 + \frac{n_2}{n_1}\right)\left(1 + \frac{n_3}{n_2}\right)\left(1 - \frac{n_2}{n_1}\right)\left(1 - \frac{n_3}{n_2}\right)2\cos(2k_2 d) \\
&= \left(1 + \frac{n_2}{n_1}\right)^2\left(1 + \frac{n_3}{n_2}\right)^2 + \left(1 - \frac{n_2}{n_1}\right)^2\left(1 - \frac{n_3}{n_2}\right)^2 \\
&\quad + 2\left[1 - \left(\frac{n_2}{n_1}\right)^2\right]\left[1 - \left(\frac{n_3}{n_2}\right)^2\right]\left[1 - 2\sin^2(k_2 d)\right] \\
&= 4\left(1 + \frac{n_2}{n_1}\frac{n_3}{n_2}\right)^2 - 4\left[1 - \left(\frac{n_2}{n_1}\right)^2\right]\left[1 - \left(\frac{n_3}{n_2}\right)^2\right]\sin^2(k_2 d) \\
&= 100 - 4[1 - 4][1 - 4]\sin^2(k_2 d) \\
&= 100 - 36\sin^2(k_2 d)
\end{aligned}
$$

or alternatively

$$
\begin{aligned}
\left|\frac{E_i}{E_t}\right|^2 &= \frac{1}{4}\left[\left(1 + \frac{n_2}{n_1}\frac{n_3}{n_2}\right)^2 - \left[1 - \left(\frac{n_2}{n_1}\right)^2\right]\left[1 - \left(\frac{n_3}{n_2}\right)^2\right]\sin^2(k_2 d)\right] \\
&= \frac{1}{4}\left[25 - [1 - 4][1 - 4]\sin^2(k_2 d)\right] \\
&= \frac{1}{4}\left[25 - 9\sin^2(k_2 d)\right]
\end{aligned}
$$

c) If the thickness d is varied, the ratio $|E_i/E_t|^2$ oscillates. What is the period of the oscillation? Assuming $n_1 < n_2 < n_3$, for which values of d is $|E_i/E_t|^2$ the smallest?

The period of the oscillation is λ_2, and $|E_i/E_t|^2$ is the smallest for $d = (2m + 1)\lambda_2/2$ with m being an integer.

Mathematical Formulae

Definitions

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\vec{E}(\vec{r}) = -\nabla\phi(\vec{r})$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\vec{J}(\vec{r}') \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\Delta\phi = -\int \vec{E}(\vec{r}) \cdot d\vec{r}$$

$$C = \frac{Q}{\Delta\phi}; \qquad \sigma = -\varepsilon_0\frac{\partial\phi}{\partial n}$$

$$\nabla\vec{E} = \frac{\rho}{\varepsilon_0}; \qquad \nabla\vec{B} = 0$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}; \qquad \nabla\times\vec{B} = \mu_0\vec{J}$$

In spherical coordinates

$$\vec{E} = -\nabla\phi(r,\theta,\phi) = -\hat{r}\frac{\partial\phi(r,\theta,\phi)}{\partial r} - \hat{\theta}\frac{1}{r}\frac{\partial\phi(r,\theta,\phi)}{\partial\theta} - \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\phi(r,\theta,\phi)}{\partial\phi}$$

Integrals, Series, Expansions and Identities

$$\int_0^{2\pi}\frac{d\varphi}{\sqrt{a-b\cos\varphi}} = \frac{1}{a-b}K\left[\frac{-2b}{a-b}\right]$$ where K is the complete elliptic integral

$$\int_0^b \frac{x^3}{[a^2+x^2]^{3/2}}dx = \frac{2a^2+b^2}{[a^2+b^2]^{1/2}} - 2a$$

$$\int \frac{1}{[a^2+x^2]^{3/2}}dx = \frac{x}{a^2[a^2+x^2]^{1/2}}$$

$$\int dr \frac{r^2}{\sqrt{z^2+r^2}} = \frac{1}{2}r\sqrt{z^2+r^2} - \frac{1}{2}z^2\ln\left[r+\sqrt{z^2+r^2}\right]$$

$$\int_0^c dx\left[\frac{2(a+x)^2+b^2}{\left[(a+x)^2+b^2\right]^{1/2}} - 2(a+x)\right] = \left[(a+c)\left(\sqrt{(a+c)^2+b^2} - (a+c)\right) - a\left(\sqrt{a^2+b^2} - a\right)\right]$$

$$\int_0^1 dx\, P_l(x) = \begin{cases} 0 & \text{for even } l \\ 1 & \text{for } l = 0 \\ (-1)^{\frac{l-1}{2}} \frac{(l+1)(l-1)!}{2^{l+1}\left[\left(\frac{l+1}{2}\right)!\right]^2} & \text{for odd } l \end{cases}$$

$$\int_{-1}^0 dx\, P_l(x) = (-1)^l \int_0^1 dx\, P_l(x)$$

$$\int_{-1}^1 dx\, P_l(x)P_m(x) = \frac{2}{2l+1}\delta_{lm}$$

$$\int_{-1}^1 dx\, [P_l(x)]^2 = \frac{2}{2l+1}$$

$$P_0(\cos\theta) = 1$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{1}{2}[3\cos^2\theta - 1]$$

$$P_3(\cos\theta) = \frac{1}{2}[5\cos^3\theta - 3\cos\theta]$$

$$\Phi(r,\theta) = \sum_n [A_n r^n + B_n r^{-(n+1)}]P_n(\cos\theta)$$

$$\int \frac{1}{r^2}dr = -\frac{1}{r}$$

$$\int \frac{1}{r}dr = \ln r$$

$$\sqrt{1+x} = 1 + \frac{x}{2} + \ldots$$