University of Illinois at Chicago
Department of Physics

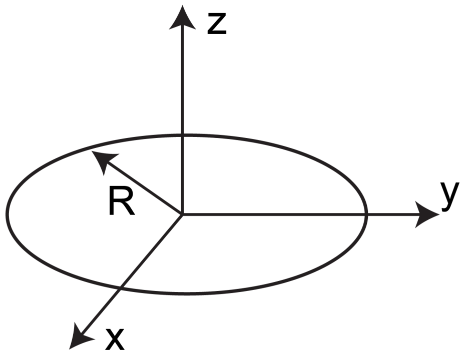
Electricity and Magnetism
Qualifying Exam

January 10, 2014
9:00am-12:00pm

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted towards the exam's total score.

Problem 1

Consider a disc of charge density $\sigma(\vec{r}) = \sigma_0 |\vec{r}|$ and radius R that lies within the xy-plane. The origin of the coordinate systems is located at the center of the disc (see figure below).



a) Give the potential at the point $\vec{P} = (0, 0, z)$ in terms of σ_0, R, and z.

b) We next put a conducting plane into the $z = d$ plane. The potential of the conducting plane is fixed at $V = 0$. Compute the total potential, ϕ_{tot}, at a point $\vec{P} = (0, 0, z)$.

c) If the total charge, Q, on the disc is fixed, find the charge density in terms of Q and use it to obtain the form of ϕ_{tot} in terms of Q, R, z in the limit $R \gg z, d$ up to leading order in (z/R).

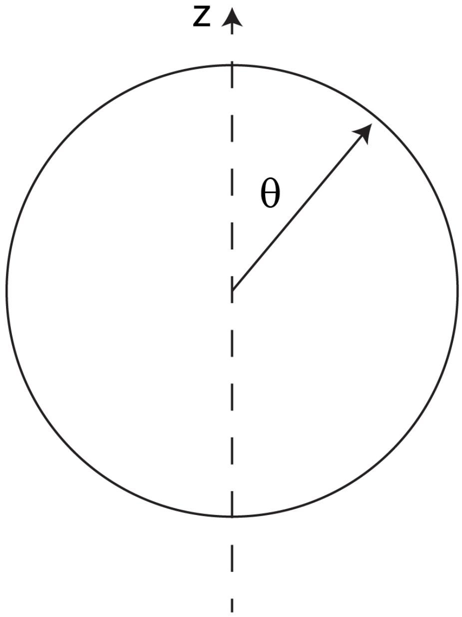
d) Give an explicit form of the induced charge density at $\vec{P} = (0, 0, d)$ in the limit $R \gg d$ using the results of part c).

Problem 2

Consider a sphere of radius R. The potential on the surface of the sphere varies as (see figure below)

$$\phi(\theta) = \phi_0 \cos^2 \theta$$

The region inside and outside the sphere is empty.



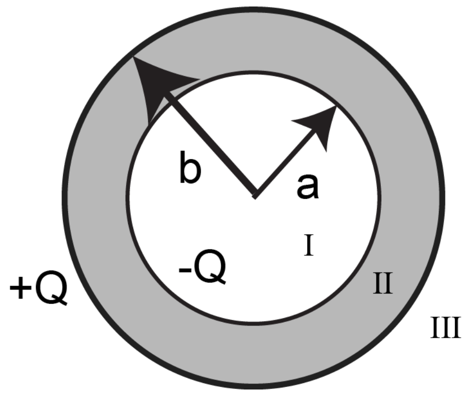
a) Compute the potential inside and outside of the sphere.

b) Compute the electric field inside the sphere.

c) Using Gauss' law, show that while the electric field inside the sphere is non-zero, no charge is contained inside the sphere.

Problem 3

a) Consider two conducting spheres with radii a and b as shown in the figure below. The volume between the two spheres (region II) is filled with a material of permitivity ε. The permitivity in regions I and III is that of free space, ε_0. The two spheres are uniformly charged with total charge $\pm Q$.

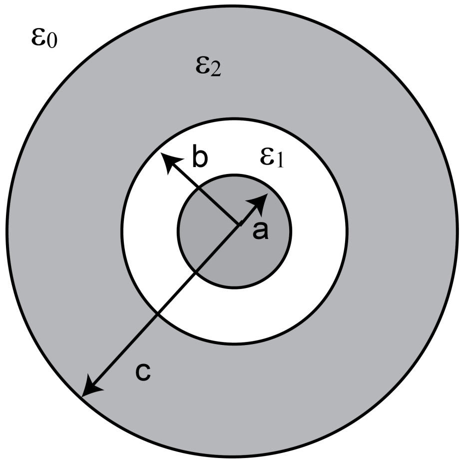


(i) Compute the magnitude and direction of the electric field in regions I, II, and III.

(ii) Compute the capacitance of the two spheres.

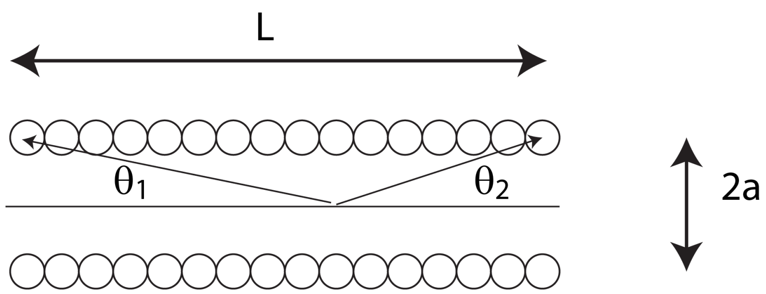
b) Consider next two infinitely long concentric cylinders, as shown in the figure below. The inner cylinder of radius a is a conductor with linear charge density $\lambda_1 > 0$. The second cylinder with inner radius b and outer radius c consists of a material with permitivity ε_2 and is uniformly charged with line charge density $\lambda_2 < 0$ ($\lambda_1 > |\lambda_2|$). The space between the two cylinders (i.e., $a < r < b$) is filled with a medium of permitivity ε_1. The medium outside the outer cylinder possesses the permitivity ε_0.

Compute the potential difference between a point at $|\vec{r}| = 2c$ (measured from the center of the inner cylinders) and the center of the inner cylinder.



Problem 4

A solenoid of finite length L and a readius a has N turns per unit length and carries a current I, with cicular cross section as shown in the figure below.

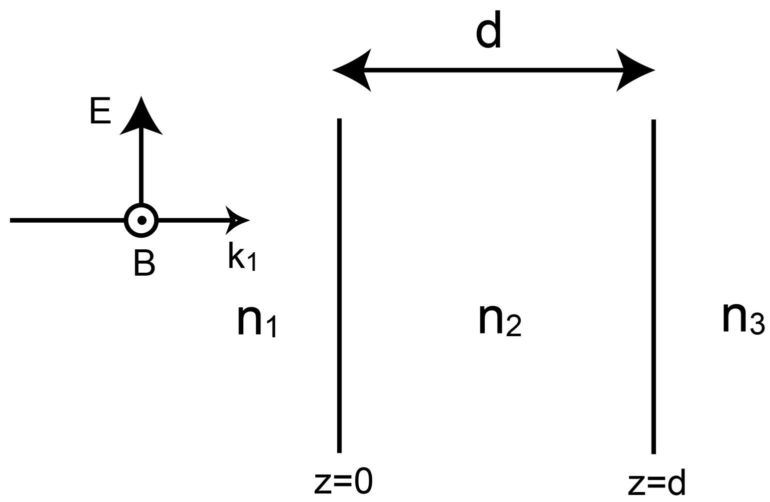


a) Compute the magnetic induction on the solenoid axis in the limit $NL \longrightarrow \infty$ in terms of the angles θ_1 and θ_2.

b) For $a \gg L$, how does the magnetic induction scale with a?

Problem 5

An electromagnetic plane wave is incident perpendicular to a layered interface, as shown in the figure below. The indices of refraction of the three media is $n_1, n_2 = 2n_1$ and $n_3 = 4n_1$ while the permeability of all three regions is the same, μ_0. The thickness of the intermediate layer is d. Each of the other media is semi-infinite.



a) State the boundary conditions at both interfaces in terms of the electric fields.

b) Compute the ratio between the incident electric field in medium 1 and the transmitted electric field in medium 3, i.e., compute $|E_i/E_t|^2$.

c) If the thickness d is varied, the ratio $|E_i/E_t|^2$ oscillates. What is the period of the oscillation? For which values of d is $|E_i/E_t|^2$ the smallest?

Mathematical Formulae

Definitions

$$\begin{aligned}
\Phi(\vec{r}) &= \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \\
\vec{E}(\vec{r}) &= -\nabla\phi(\vec{r}) \\
\vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi}\int d^3r'\, \vec{J}(\vec{r}') \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \\
\vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi}\int d^3r'\, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}
\end{aligned}$$

$$\Delta\phi = -\int \vec{E}(\vec{r}) \cdot d\vec{r}$$

$$C = \frac{Q}{\Delta\phi}; \qquad \sigma = -\varepsilon_0\frac{\partial\phi}{\partial n}$$

$$\begin{aligned}
\nabla\vec{E} &= \frac{\rho}{\varepsilon_0}; \qquad \nabla\vec{B} = 0 \\
\nabla\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}; \qquad \nabla\times\vec{B} = \mu_0\vec{J}
\end{aligned}$$

In spherical coordinates

$$\vec{E} = -\nabla\phi(r,\theta,\phi) = -\hat{r}\frac{\partial\phi(r,\theta,\phi)}{\partial r} - \hat{\theta}\frac{1}{r}\frac{\partial\phi(r,\theta,\phi)}{\partial\theta} - \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\phi(r,\theta,\phi)}{\partial\phi}$$

Integrals, Series, Expansions and Identities

$$\int_0^{2\pi}\frac{d\varphi}{\sqrt{a-b\cos\varphi}} = \frac{1}{a-b}K\left[\frac{-2b}{a-b}\right]$$ where K is the complete elliptic integral

$$\begin{aligned}
\int_0^b \frac{x^3}{[a^2+x^2]^{3/2}}dx &= \frac{2a^2+b^2}{[a^2+b^2]^{1/2}} - 2a \\
\int \frac{1}{[a^2+x^2]^{3/2}}dx &= \frac{x}{a^2\,[a^2+x^2]^{1/2}}
\end{aligned}$$

$$\int dr\frac{r^2}{\sqrt{z^2+r^2}} = \frac{1}{2}r\sqrt{z^2+r^2} - \frac{1}{2}z^2\ln\left[r+\sqrt{z^2+r^2}\right]$$

$$\int_0^c dx \left[\frac{2\,(a+x)^2 + b^2}{\left[(a+x)^2 + b^2\right]^{1/2}} - 2\,(a+x) \right] = \left[(a+c) \left(\sqrt{(a+c)^2 + b^2} - (a+c) \right) - a \left(\sqrt{a^2+b^2} - a \right) \right]$$

$$\begin{aligned} \int_0^1 dx\ P_l(x) \ &= \ \begin{cases} 0 & \text{for even } l \\ 1 & \text{for } l = 0 \\ (-1)^{\frac{l-1}{2}} \frac{(l+1)(l-1)!}{2^{l+1}\left[\left(\frac{l+1}{2}\right)!\right]^2} & \text{for odd } l \end{cases} \\ \int_{-1}^0 dx\, P_l(x) \ &= \ (-1)^l \int_0^1 dx\ P_l(x) \\ \int_{-1}^1 dx\, P_l(x) P_m(x) \ &= \ \frac{2}{2l+1} \delta_{lm} \end{aligned}$$

$$\int_{-1}^1 dx\ \ [P_l(x)]^2 = \frac{2}{2l+1}$$

$$\begin{aligned} P_0(\cos\theta) \ &= \ 1 \\ P_1(\cos\theta) \ &= \ \cos\theta \\ P_2(\cos\theta) \ &= \ \frac{1}{2}\left[3\cos^2\theta - 1\right] \\ P_3(\cos\theta) \ &= \ \frac{1}{2}\left[5\cos^3\theta - 3\cos\theta\right] \end{aligned}$$

$$\Phi(r,\theta) = \sum_n \left[A_n r^n + B_n r^{-(n+1)}\right] P_n\left(\cos\theta\right)$$

$$\begin{aligned} \int \frac{1}{r^2} dr \ &= \ -\frac{1}{r} \\ \int \frac{1}{r} dr \ &= \ \ln r \end{aligned}$$

$$\sqrt{1+x} = 1 + \frac{x}{2} + ...$$