

UNIVERSITY OF ILLINOIS AT CHICAGO
DEPARTMENT OF PHYSICS

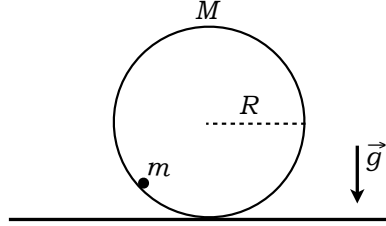
Classical Mechanics
Ph.D. Qualifying Examination

9 January, 2014
9:00 to 12:00

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted toward the exams total score.

Problem 1

A thin circular loop of radius R , with mass M distributed uniformly along its circumference, is free to roll along a horizontal surface without slipping. A point particle of mass m is attached to the inside of the loop and is constrained to slide along the inside perimeter of the loop without friction. The system is in a uniform gravitational acceleration g .



(a) Write down the Lagrangian for this system.

Solution: Let X be the generalised coordinate for the loop and θ be the generalised coordinate for the particle. Then $T_{\text{loop}} = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}I\omega^2$, where $I = MR^2$ and $\dot{X} = R\omega$ (rolls without slipping). Hence, $T_{\text{loop}} = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}(MR^2)(\frac{\dot{X}}{R})^2 = M\dot{X}^2$. Temporarily, use (x, y) coordinates for $T_{\text{particle}} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$. Re-expressing this in terms of our generalised coordinates, we get $x = X + R\sin\theta$ and $y = R(1 - \cos\theta)$, so that $\dot{x} = \dot{X} + R\dot{\theta}\cos\theta$ and $\dot{y} = R\dot{\theta}\sin\theta$. Substituting, we have $T_{\text{particle}} = \frac{1}{2}m[(\dot{X} + R\dot{\theta}\cos\theta)^2 + R^2\dot{\theta}^2\sin^2\theta] = \frac{1}{2}m[\dot{X}^2 + 2R\dot{X}\dot{\theta}\cos\theta + R^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta)] = \frac{1}{2}m[\dot{X}^2 + 2R\dot{X}\dot{\theta}\cos\theta + R^2\dot{\theta}^2]$. Also, $V_{\text{particle}} = mgy = mgR(1 - \cos\theta)$.

$$\text{Hence } L = T - V = M\dot{X}^2 + \frac{1}{2}m[\dot{X}^2 + 2R\dot{X}\dot{\theta}\cos\theta + R^2\dot{\theta}^2] - mgR(1 - \cos\theta)$$

(b) Find the equations of motion and any possible equilibrium positions for the particle.

Solution: The EOMs are given by $\frac{d}{dt}\frac{\partial L}{\partial \dot{X}} = \frac{\partial L}{\partial X}$, which implies that $\frac{d}{dt}[(2M + m)\dot{X} + mR\dot{\theta}\cos\theta] = 0$.

$$(*) \quad \text{Hence } (2M + m)\ddot{X} + mR[\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta] = 0.$$

Also, $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$, which implies that $\frac{d}{dt}[mR\dot{X}\cos\theta + mR^2\dot{\theta}] = -mgR\sin\theta - mR\dot{X}\dot{\theta}\sin\theta$. Continuing, $mR\ddot{X}\cos\theta - mR\dot{X}\dot{\theta}\sin\theta + mR^2\ddot{\theta} = -mgR\sin\theta - mR\dot{X}\dot{\theta}\sin\theta$, which implies that:

$$(**) \quad mR\ddot{X}\cos\theta + mR^2\ddot{\theta} = -mgR\sin\theta.$$

We can obtain the EOM for θ alone. From (*), we have: $\ddot{X} = \frac{mR(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta)}{2M + m}$, and inserting into (**), we obtain:

$$(***) \quad R\cos\theta\frac{mR(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta)}{2M + m} + R^2\ddot{\theta} + gR\sin\theta = 0$$

The requirement for equilibrium is that $\dot{\theta} = 0$ and $\ddot{\theta} = 0$. Inserting into (***), we get $0 + 0 + gR\sin\theta = 0$. Solving for θ , we find that

$$\theta = 0 \text{ or } \theta = \pi$$

(c) Which of the equilibrium positions are stable and which are unstable (you may qualitatively answer this part, if you wish)?

Solution: By inspection, $\theta = 0$ corresponds to the bottom of the loop and is therefore a stable point of equilibrium. Also $\theta = \pi$ corresponds to the top of the loop and therefore is a point of unstable equilibrium.

(d) Find the frequency of small amplitude oscillations of the particle about all possible positions of *stable* equilibrium. Consider your results in the limit $M \gg m$ and discuss.

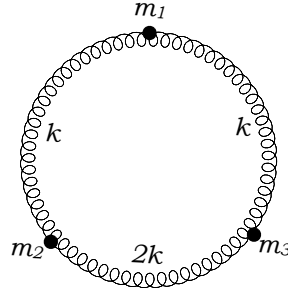
Solution: Consider a small perturbation ϵ about $\theta = 0$. Then $\theta \approx \epsilon$; $\dot{\theta} \approx \dot{\epsilon}$; $\ddot{\theta} \approx \ddot{\epsilon}$ and to 1st order in (**), we have $R(1 - \frac{mR(\dot{\epsilon} - \ddot{\epsilon})}{2M+m}) + R^2\ddot{\epsilon} + gR\epsilon = 0$. To first order, we can ignore $\dot{\epsilon} \approx 0$. Hence, $-R^2\frac{m}{2M+m}\ddot{\epsilon} + R^2\ddot{\epsilon} + gR\epsilon = 0$.

Continuing, we have $(1 - \frac{m}{2M+m})R^2\ddot{\epsilon} + gR\epsilon = 0$, which implies that $\omega = \sqrt{\frac{g}{(1 - \frac{m}{2M+m})R}}$. Taking the limit

$M \gg m$, we get: $\omega = \sqrt{\frac{g}{R}}$ which is just the frequency of a simple pendulum.

Problem 2

Three particles of equal mass $m = m_1 = m_2 = m_3$ are constrained to slide around a frictionless circular track and are connected by three equal length springs that have spring constants k , k , and $2k$, respectively. See the below diagram. Initially, the system is in equilibrium and the springs are initially without any tension (i.e. they are neither stretched nor compressed).



(a) Write down the Lagrangian. Obtain the Equations of Motion (EOM) for this normal mode problem and express the EOM as an eigenvalue equation: $\Omega|\omega\rangle = \omega^2|\omega\rangle$, where Ω is a 3×3 matrix and ω is the oscillation frequency of a normal mode.

Solution: Choose generalised coordinates x_i with $i = 1, 2, 3$, representing the arc-length of particle i from its equilibrium position around the perimeter of the circle. Then $T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2$ and

$V = \frac{1}{2}k(x_1 - x_2)^2 + \frac{1}{2}(2k)(x_2 - x_3)^2 + \frac{1}{2}k(x_3 - x_1)^2$; the Lagrangian is just $L = T - V$. To arrive at the eigenvalue equation, we re-express the Lagrangian in matrix notation. Let $|x\rangle = (x_1, x_2, x_3)$. Then $T = \frac{1}{2}m\langle\dot{x}|\dot{x}\rangle$. Expanding out V , we get $V = \frac{1}{2}k(2x_1^2 + 3x_2^2 + 3x_3^2 - x_1x_2 - x_2x_1 - 2x_2x_3 - 2x_3x_2 - x_3x_1 - x_1x_3)$. Identifying indices i, j with columns and rows, we write $V = \frac{1}{2}k\langle x|$

$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{pmatrix} |x\rangle \equiv \frac{1}{2}\langle x|K|x\rangle$. Hence,

the Lagrangian can be rewritten as $L = \frac{1}{2}m\langle\dot{x}|\dot{x}\rangle - \frac{1}{2}\langle x|K|x\rangle$. The equations of motion are determined by:

$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$, which leads to $m|\ddot{x}\rangle + K|x\rangle = 0$ (*). Assume solutions of the form $|x\rangle = e^{i\omega t}|x\rangle$; then $|\ddot{x}\rangle = -\omega^2|x\rangle$. Substituting these into (*) and dividing by m , we get $-\omega^2|x\rangle + \frac{1}{m}K|x\rangle = 0$, or $\omega^2|x\rangle = \frac{1}{m}K|x\rangle \equiv \Omega|x\rangle$, which is an eigenvalue equation. The eigenvector solutions correspond to specific values of ω , which are normal modes: $|x\rangle = |\omega\rangle$. Hence, the eigenvalue equation for normal modes is just $\Omega|\omega\rangle = \omega^2|\omega\rangle$.

(b) Show by direct substitution into the eigenvalue equation in part (a) above, that there is a zero-frequency normal mode associated with an overall “rotation” of the system. Write the corresponding normalised eigenvector $|\omega_1\rangle$.

Solution: An overall rotation of the system by x means $x = x_1 = x_2 = x_3$, or $|x\rangle = x(1, 1, 1)$. Normalising:

$\frac{1}{\sqrt{3}}(1, 1, 1) \equiv |\omega_1\rangle$. Inserting in part (a), we get: $\Omega|\omega_1\rangle = \frac{k}{m} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0|\omega_1\rangle$,

which implies that $\omega_1 = 0$, which is a zero-frequency normal mode.

(c) Considering the symmetry of the configuration, write down another normal mode unit-vector $|\omega_2\rangle$ for the case when particle 1, opposite the spring with spring constant $2k$, remains in a fixed position. Using direct substitution into the eigenvalue equation of (a), check that $|\omega_2\rangle$ is indeed an eigenvector of Ω .

Solution: Considering symmetry and the hint, another normal mode occurs when the mass opposite the spring with constant $2k$ is fixed, while the other two masses move in opposite directions with equal amplitude: $|x\rangle = x(0, 1, -1)$. Normalising: $\frac{1}{\sqrt{2}}(0, -1, 1) \equiv |\omega_2\rangle$. Substituting this into the eigenvalue equation

of part (a), $\Omega|\omega_2\rangle = \frac{k}{m} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \left(\frac{5k}{m}\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \sqrt{\frac{5k}{m}}|\omega_2\rangle$, so that $\omega_2 = \sqrt{\frac{5k}{m}}$.

(d) Use $|\omega_1\rangle$ and $|\omega_2\rangle$ (or any other method you wish) to determine the final normal mode unit-vector $|\omega_3\rangle$. Via direct substitution into the eigenvalue equation of (a), show that $|\omega_3\rangle$ is also an eigenvector of Ω .

Solution: The final normal mode can be obtained by forming the cross product of $|\omega_1\rangle$ with $|\omega_2\rangle$:

$$\begin{vmatrix} |x_1\rangle & |x_2\rangle & |x_3\rangle \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \frac{1}{N} \begin{pmatrix} 1 - (-1) \\ -1 + 0 \\ -(1 + 0) \end{pmatrix}. \text{ Hence } |\omega_3\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}. \text{ Substituting this into the eigenvalue equation of part (a), } \Omega|\omega_3\rangle = \frac{k}{m} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \left(\frac{3k}{m}\right) \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \sqrt{\frac{3k}{m}}|\omega_3\rangle, \text{ so that } \omega_3 = \sqrt{\frac{3k}{m}}.$$

(e) Find the resulting time-dependent motion of the system if particle 1 is initially displaced from its equilibrium position by an amount x_0 . You may leave your answer in terms of ω_1 , ω_2 , and ω_3 , if you wish.

Solution: The most general solution is a superposition of the normal mode solutions:

$$|x(t)\rangle = (A_1 + B_1 t) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (A_2 \cos \omega_2 t + B_2 \sin \omega_2 t) \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + (A_3 \cos \omega_3 t + B_3 \sin \omega_3 t) \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$

Since all particles are initially at rest, $B_1 = 0$. Without loss of generality, we can choose phases so that we only have cosine solutions. Thus $B_2 = B_3 = 0$. Hence at $t = 0$:

$$|x(0)\rangle = \begin{pmatrix} x_0 \\ 0 \\ 0 \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + A_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + A_3 \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$

So, $x_0 = A_1 + 2A_3$ (*), $0 = A_1 - A_2 - A_3$ (**), and finally $0 = A_1 + A_2 - A_3$ (***). Adding (**) + (***), we get $0 = 2A_1 - 2A_3$ which implies $A_1 = A_3$. Subtracting (**) - (***), we get $0 = -2A_2$. Inserting everything in (1), we get $x_0 = 3A_1$, so $A_1 = \frac{x_0}{3}$.

Hence, $|x(t)\rangle = \frac{x_0}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{x_0}{3} \cos(\omega_3 t) \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$, where $\omega_3 = \sqrt{\frac{3k}{m}}$

Problem 3

A ball of putty of mass m travels at speed v towards another ball of putty, also of mass m , which is at rest (but which is free to move) in the Lab frame of reference. They “collide” and “stick” together forming a new object.

(a) Assume that the collision is “head-on” and the speed v of the incoming ball is *non-relativistic*, i.e. much smaller than c the speed of light. Determine the final speed V and mass M of the new non-relativistic object after the collision.

Solution:

$$\text{Before the collision, } \vec{p}_1 = \begin{pmatrix} mv \\ 0 \\ 0 \end{pmatrix}, E_1 = \frac{1}{2}mv^2; \quad \vec{p}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, E_2 = 0.$$

$$\text{After the collision, } p_{\text{after}} = \begin{pmatrix} (2m)V \\ 0 \\ 0 \end{pmatrix}, E_{\text{after}} = \frac{1}{2}(2m)V^2 = mV^2.$$

Conservation of 3-momentum implies $\vec{p}_{\text{before}} = \vec{p}_1 + \vec{p}_2 = \vec{p}_{\text{after}}$. Hence, $mv = (2m)V$, which implies $V = \frac{1}{2}v$. The mass M of the new object is just $M = 2m$.

(b) Again assume that the collision is “head-on”, but now assume that the speed v of the incoming ball is *relativistic* such that it can not be neglected compared with c , the speed of light. Determine the final speed V and rest mass M of the new relativistic object after the collision.

Solution: Let's initially work in units where $c = 1$. Then $\beta = \frac{v}{c} = v$ and $\beta' = \frac{V}{c} = V$.

$$\text{Before the collision } p_1^\mu = \begin{pmatrix} m\gamma \\ m\beta\gamma \\ 0 \\ 0 \end{pmatrix} \text{ and } p_2^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ where } \gamma = 1/\sqrt{1-\beta^2}.$$

$$\text{After the collision } p_{\text{after}}^\mu = \begin{pmatrix} M\gamma' \\ M\beta'\gamma' \\ 0 \\ 0 \end{pmatrix}, \text{ where } \gamma' = 1/\sqrt{1-\beta'^2}.$$

Conservation of 4-momentum implies $p_{\text{before}}^\mu = p_1^\mu + p_2^\mu = p_{\text{after}}^\mu$. Hence, $m\gamma + m = M\gamma'$ (*) and $m\beta\gamma + 0 = M\beta'\gamma'$ (**). Dividing (*) by (**), we get: $\frac{m(\gamma+1)}{m\beta\gamma} = \frac{M\gamma'}{M\beta'\gamma'}$. This implies that: $V = \beta' = \frac{\beta\gamma}{\gamma+1}$.

Restoring c , we get $V = \frac{v/\sqrt{1-(v/c)^2}}{1+1/\sqrt{1-(v/c)^2}} = \frac{v}{1+\sqrt{1-(v/c)^2}}$. The rest mass M is just $M^2 = p_{\text{after}}^2 = p_{\text{before}}^2 = (m\gamma+m)^2 - (m\beta\gamma)^2 = m^2(\gamma+1)^2 - m^2\beta^2\gamma^2 = m^2(\gamma^2 + 2\gamma + 1 - \beta^2\gamma^2) = m^2(1+2\gamma+\gamma^2[1-\beta^2]) = m^2(2+2\gamma)$.

Hence, $M = m\sqrt{2(\gamma+1)}$

(c) Compare the non-relativistic case (a) with the relativistic case (b) by filling in the below table with “Yes” or “No” answers.

	non-Relativistic case	Relativistic case
Is total kinetic energy conserved?		
Is total mass conserved?		

For any “No” answer you provide, explicitly show that the observable is not-conserved, and qualitatively explain why the observable is larger (or smaller) before versus after the collision.

Solution:

	non-Relativistic case	Relativistic case
Is total kinetic energy conserved?	No! (*)	Yes!
Is total mass conserved?	Yes!	No! (**)

(*) $[\frac{1}{2}mv^2]_{\text{before}} > [\frac{1}{2}m(\frac{v}{2})^2]_{\text{after}}$, so the kinetic energy after the collision is always smaller; the decrease reflects the fact that some of the kinetic energy is converted into heat and, classically, it is “lost” (i.e. not counted as kinetic energy).

(**) $[2m]_{\text{before}} < [m\sqrt{2(\gamma+1)}]_{\text{after}}$, so the rest mass after the collision is always larger ($v > 0 \rightarrow \gamma > 1$); the increase reflects the fact that heat is generated after the inelastic collision, which is relativistically converted into mass.

(d) Reconsider the *non-relativistic* case, but now assume that there is an impact parameter r between the two balls of putty and that the two balls are instantaneously captured into a bound system by an infinitely stiff, massless string. Assume that the diameter of the balls are much smaller than r , so that they can be considered point-like particles – that is, assume that the new combined object after the collision consists of two point particles of mass m , separated by a fixed distance r . Determine the final speed and mass of the new combined object after the “collision.” Is total kinetic energy conserved? Is mass conserved?

Solution:

Before the collision, $\vec{p}_1 = \begin{pmatrix} mv \\ 0 \\ 0 \end{pmatrix}$, $\vec{p}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

After the collision, $\vec{p}_{\text{after}} = \begin{pmatrix} 2mV \\ 0 \\ 0 \end{pmatrix}$; $I_{\text{cm}} = 2[m(\frac{r}{2})^2] = \frac{1}{2}mr^2$.

As in part (b), conservation of 3-momentum leads to $V = \frac{1}{2}v$, which corresponds to the velocity of the CM frame of reference. One can use conservation of angular momentum, or just recognise that after the collision (in the CM frame of reference) $\omega_{\text{cm}} = \frac{v/2}{r/2} = \frac{v}{r}$. Now $T_{\text{linear}} = \frac{1}{2}(m+m)V^2 = \frac{1}{2}(2m)(\frac{v}{2})^2 = \frac{1}{4}mv^2$ and $T_{\text{rot}} = \frac{1}{2}I_{\text{cm}}\omega_{\text{cm}}^2 = \frac{1}{2}(\frac{1}{2}mr^2)(\frac{v}{r})^2 = \frac{1}{4}mv^2$. Hence, $T_{\text{after}} = T_{\text{linear}} + T_{\text{rot}} = \frac{1}{2}mv^2 = T_{\text{before}}$. Also $M_{\text{before}} = 2m = M_{\text{after}}$. Thus, total kinetic energy is conserved, and total mass is conserved! (Aside: There are no sudden changes in velocities to any of the particles, so the amount of linear kinetic energy which is “lost” after the collision is continuously converted into rotational kinetic energy.)

Problem 4

An idealised rigid body consists of three equal-mass points (of unit mass) that are placed in a force-free environment. The coordinates of the mass points are given by:

$$|r_1\rangle = \begin{pmatrix} 0 \\ \sqrt{22} \\ 0 \end{pmatrix} \quad |r_2\rangle = \begin{pmatrix} -\frac{3}{\sqrt{2}} \\ -\sqrt{\frac{11}{2}} \\ 0 \end{pmatrix} \quad |r_3\rangle = \begin{pmatrix} \frac{3}{\sqrt{2}} \\ -\sqrt{\frac{11}{2}} \\ 0 \end{pmatrix}$$

(a) Find the principle axes of the rigid body.

Solution: Upon inspection, the coordinates are given with respect to the centre of mass (CM). Since there are only three masses, they define a triangle lying entirely in the (x, y) plane. Thus, one principle axis points along the semi-major axis of the triangle, i.e. it points in the direction of $|r_1\rangle$. Hence, $|I_1\rangle = (0, 1, 0)$. Another principle axis points along the semi-minor axis of the triangle, i.e. it points along the direction from $|r_2\rangle$ to $|r_3\rangle$. Hence, $|I_2\rangle = \frac{1}{N}(|r_3\rangle - |r_2\rangle) = \frac{1}{N}(\frac{6}{\sqrt{2}}, 0, 0) = (1, 0, 0)$. The final principle axis corresponds to an in-plane rotation of the triangle about the CM and is thus perpendicular to $|I_1\rangle$ and $|I_2\rangle$. Hence,

$$|I_3\rangle = \frac{1}{N} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \frac{1}{N}(\hat{i}[0+0] + \hat{j}[0+0] + \hat{k}[0-1]) = (0, 0, 1).$$

(b) In the CM frame of reference, find the moments of inertia, I_1, I_2, I_3 , corresponding to the principle axes in part (a). Define the (normalised) principal axes $|I_1\rangle, |I_2\rangle$, and $|I_3\rangle$ in such a way that $I_1 < I_2 < I_3$.

Solution: We now find the moments of inertia (for unit masses), defined to be $I_i = \sum_{\alpha=1}^3 [\langle r_\alpha | r_\alpha \rangle - \langle r_\alpha | I_i \rangle^2]$. From part (a), we see that the coordinate system (x, y, z) is already aligned with the principle axes such that $|I_1\rangle = |\hat{y}\rangle, |I_2\rangle = |\hat{x}\rangle, |I_3\rangle = |\hat{z}\rangle$. Hence, the most straight forward calculation is to simply plug the corresponding inner products into the definition for I_i above. However, a slightly quicker way forward is to use the vector identity $(\vec{A} \times \vec{B})^2 = A^2 B^2 - (\vec{A} \cdot \vec{B})^2$ so that $I_i = \sum_{\alpha=1}^3 (|r_\alpha \times |I_i\rangle)^2$. Thus:

$$\begin{aligned} I_1 &= (|r_1\rangle \times |\hat{y}\rangle)^2 + (|r_2\rangle \times |\hat{y}\rangle)^2 + (|r_3\rangle \times |\hat{y}\rangle)^2 \\ &= 0 + \frac{9}{2} + \frac{9}{2} = 9 \\ I_2 &= (|r_1\rangle \times |\hat{x}\rangle)^2 + (|r_2\rangle \times |\hat{x}\rangle)^2 + (|r_3\rangle \times |\hat{x}\rangle)^2 \\ &= 22 + \frac{11}{2} + \frac{11}{2} = 33 \\ I_3 &= (|r_1\rangle \times |\hat{z}\rangle)^2 + (|r_2\rangle \times |\hat{z}\rangle)^2 + (|r_3\rangle \times |\hat{z}\rangle)^2 \\ &= 22 + \frac{9}{2} + \frac{11}{2} + \frac{9}{2} + \frac{11}{2} = 42 \end{aligned}$$

(c) Show that a sufficient condition for a steady rotation of the rigid body is for the angular velocity to be along one of the principal axes of the body.

Solution: Assume $|\omega\rangle = \omega |I_i\rangle$. Then, $|L\rangle = I|\omega\rangle = I(\omega |I_i\rangle) = \omega(I |I_i\rangle) = \omega(I_i |I_i\rangle) = I_i(\omega |I_i\rangle) = I_i |\omega\rangle$. Hence $|\omega\rangle$ is an eigenvector, which implies that $|\omega\rangle = \text{constant}$. Hence, the torque

$$|\tau\rangle = \left[\frac{d|L\rangle}{dt}\right]_{\text{fixed}} = \left[\frac{d|L\rangle}{dt}\right]_{\text{rotating}} + |\omega\rangle \times |L\rangle = \left[\frac{d}{dt}(I_i |\omega\rangle)\right]_{\text{rotating}} = 0.$$

Hence, no torque will act on the body if the angular velocity is along one of the principle axes, and the rotation will be steady.

(d) Show that the steady rotation about the $|I_1\rangle$ axis is guaranteed to be stable against small perturbations. That is, write down the Equations of Motion for the rigid body and assume that the angular velocity about the $|I_1\rangle$ axis contains infinitesimal components along the other two principal axes, $|I_2\rangle$, and $|I_3\rangle$; then work to leading order in those infinitesimal quantities.

Solution: The condition for steady rotation is $|\tau\rangle = \left[\frac{d|L\rangle}{dt}\right]_{\text{fixed}} = \left[\frac{d|L\rangle}{dt}\right]_{\text{rotating}} + |\omega\rangle \times |L\rangle = 0$. Projecting along the principle axes, we get: $\left[\frac{dL_i}{dt}\right]_{\text{fixed}} = I_i\dot{\omega}_i + \epsilon_{ijk}\omega_j L_k = I_i\dot{\omega}_i + \epsilon_{ijk}I_k\omega_j\omega_k$, or

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$$

$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1$$

$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2$$

Assume a rotation about the $|I_1\rangle$ axis with a small ϵ perturbation about $|I_2\rangle$ and $|I_3\rangle$:

$|\omega\rangle = \omega_1|I_1\rangle + \omega_2|I_2\rangle + \omega_3|I_3\rangle$ with $\omega_2(0) = \omega_3(0) = \epsilon \ll \omega$. Then, $I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 \approx 0$, which implies $\omega_1 = \omega(0) = \omega = \text{const.}$ Hence $\dot{\omega}_2 = \left[\frac{I_3 - I_1}{I_2}\omega\right]\omega_3 \equiv \alpha\omega_3$ and $\dot{\omega}_3 = \left[\frac{I_1 - I_2}{I_3}\omega\right]\omega_2 \equiv \beta\omega_2$. To decouple, take the 2nd derivative: $\ddot{\omega}_2 = \alpha\dot{\omega}_3$; $\ddot{\omega}_3 = \beta\dot{\omega}_2$. Hence, $\dot{\omega}_3 = \frac{1}{\alpha}\ddot{\omega}_2 = \beta\omega_2$, so that $\ddot{\omega}_2 - \alpha\beta\omega_2 = 0$. Likewise $\dot{\omega}_2 = \frac{1}{\beta}\ddot{\omega}_3 = \alpha\omega_3$, so that $\ddot{\omega}_3 - \beta\alpha\omega_3 = 0$. Now, $\alpha\beta = \left(\frac{I_3 - I_1}{I_2}\right)\left(\frac{I_1 - I_2}{I_3}\right) = \left(\frac{42 - 9}{33}\right)\left(\frac{9 - 33}{42}\right) = -\frac{24}{42} = -\frac{4}{7}$. Hence,

$\ddot{\omega}_2 + \frac{4}{7}\omega_2 = 0$ and $\ddot{\omega}_3 + \frac{4}{7}\omega_3 = 0$, which is simple harmonic motion about both the $|I_2\rangle$ and $|I_3\rangle$ axes.

Hence the rotation about $|I_1\rangle$ is both steady and stable!

Problem 5

A particle of mass m moves under the influence of the central potential:

$$V(r) = -k/r^4$$

(a) Show that the motion occurs in a plane. Hence, use polar coordinates to write the Lagrangian for the system. Determine all constants of the motion.

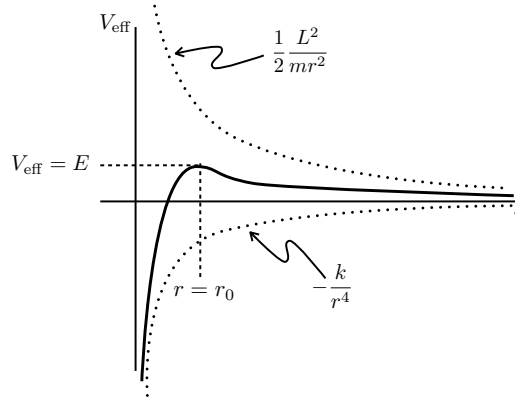
Solution: We are given that the potential is central. Hence $\vec{F}(r) = -\vec{\nabla}V(r) = -\hat{r}V'(r)$. Angular momentum is therefore conserved, since $\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \dot{\vec{r}} \times m\dot{\vec{r}} + \vec{r} \times \vec{F} = \vec{r} \times (-\hat{r})V'(r) = 0$. If we define \hat{z} to point along \vec{L} , then the motion takes place in the (\hat{x}, \hat{y}) plane. QED.

The Lagrangian is then $L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\phi})^2 - [-\frac{k}{r^4}]$. The EOM for ϕ is $\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} = 0$, hence $\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} = L = \text{const.}$ (Script L is the angular momentum.) Further, since $\frac{\partial L}{\partial t} = 0$, the total energy is a constant: $E = \frac{1}{2}m\dot{r}^2 + V(r) = \text{const.}$

(b) Make a plot of the effective potential for the radial motion of the particle. Give the general condition for a circular orbit. Does the above potential support circular orbits? If so, determine them in terms of constants of the motion; are they stable?

Solution: The effective potential is $E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - \frac{k}{r^4}$, or in terms of the angular momentum (constant of the motion), $E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2[\frac{L}{mr^2}]^2 - \frac{k}{r^4}$. Hence, the effective potential for the radial motion is:

$$V_{\text{eff}} = \frac{1}{2}\frac{L^2}{mr^2} - \frac{k}{r^4}.$$



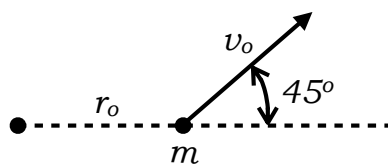
The general condition for a circular orbit is $\frac{dV_{\text{eff}}}{dr}|_{r=r_c} = 0$.

Yes, the potential has one circular orbit where the potential has zero slope:

$$\frac{dV_{\text{eff}}}{dr}|_{r=r_c} = 0 = \frac{d}{dr}\left[\frac{1}{2}\frac{L^2}{mr^2} - \frac{k}{r^4}\right]_{r=r_c} = \frac{1}{2}\left(-2\frac{L^2}{mr_c^3}\right) - \left(-4\frac{k}{r_c^5}\right). \text{ Thus, } r_c = \frac{2\sqrt{mk}}{L}.$$

No, because the circular orbit is at the maximum of the potential, the circular orbit is unstable.

(c) At time $t = 0$ the particle is at $r = r_0$ and is moving with a velocity $v = v_0$ directed at an angle of 45° with respect to the radial outward direction.



Write down the condition for the particle to escape to infinity. From these initial conditions, calculate the minimum value v_0 for which the particle is guaranteed to escape to infinity. You may work in and leave your answer in units for which $k = m = 1$.

Solution:

If $E > V_{\text{eff}}(r = r_c)$, then the particle will escape due to the centrifugal barrier of the effective potential.

From part (b), the general case for a circular orbit of arbitrary angular momentum is $r_c = \frac{2\sqrt{mk}}{L}$ or, working in units with $k = m = 1$, we have $r_c = \frac{2}{L}$. Given the initial conditions, $L = mr^2\dot{\phi} = r^2\dot{\phi} = r_0^2\dot{\phi}_0 = r_0(r_0\dot{\phi}_0) = r_0(\frac{1}{\sqrt{2}}v_0) = \frac{1}{\sqrt{2}}r_0v_0$. So, the particular circular orbit corresponding to our given angular momentum is $r_c^2 = \frac{8}{v_0^2r_0^2}$. Now from part (b), the value of the effective potential at the radius of the circular orbit is (in our units) $V_{\text{eff}}(r_c) = \frac{1}{2}\frac{L^2}{r_c^2} - \frac{1}{r_c^4} = \frac{1}{2}\frac{r_0^2v_0^2}{2r_c^2} - \frac{1}{r_c^4} = \frac{1}{2}\frac{r_0^2v_0^2}{2\frac{8}{v_0^2r_0^2}} - \frac{v_0^4r_0^4}{64} = \frac{r_0^4v_0^4}{32} - \frac{v_0^4r_0^4}{64} = \frac{r_0^4v_0^4}{64}$. Also, from part (a), the total energy for our initial conditions is (in our units), $E = \frac{1}{2}v_0^2 - \frac{1}{r_0^4}$. We can now require $E > V_{\text{eff}}(r_c)$ so that the particle will escape. That is, we require: $\frac{1}{2}v_0^2 - \frac{1}{r_0^4} > \frac{r_0^4v_0^4}{64}$. Rearranging, we can rewrite the inequality as $32r_0^4v_0^2 - 64 > r_0^8v_0^4$, which is just $(r_0^4v_0^2)^2 - 32(r_0^4v_0^2) + 64 < 0$. Define $x \equiv (r_0^4v_0^2)$, then $x^2 - 32x + 64 < 0$, which has two solutions $\frac{32 \pm \sqrt{(32)^2 - 4(1)(64)}}{2(1)} = 16 \pm 8\sqrt{3}$. We want the minimum value of v_0 for the particle to escape, hence $x = r_0^4v_0^2 > 16 - 8\sqrt{3}$, or $v_0 > \frac{1}{r_0^4}\sqrt{16 - 8\sqrt{3}}$