UNIVERSITY OF ILLINOIS AT CHICAGO
DEPARTMENT OF PHYSICS

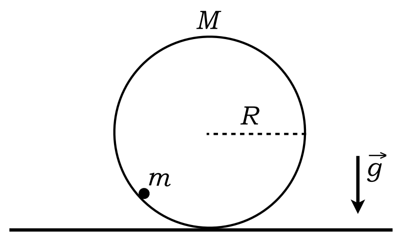
Classical Mechanics
Ph.D. Qualifying Examination

9 January, 2014
9:00 to 12:00

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted toward the exams total score.

Problem 1

A thin circular loop of radius R, with mass M distributed uniformly along its circumference, is free to roll along a horizontal surface without slipping. A point particle of mass m is attached to the inside of the loop and is constrained to slide along the inside perimeter of the loop without friction. The system is in a uniform gravitational acceleration g.



(a) Write down the Lagrangian for this system.

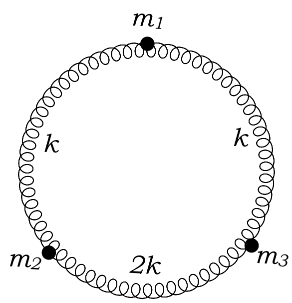
(b) Find the equations of motion and any possible equlibrium positions for the particle.

(c) Which of the equilibrium positions are stable and which are unstable (you may qualitatively answer this part, if you wish)?

(d) Find the frequency of small amplitude oscillations of the particle about all possible positions of *stable* equilibrium. Consider your results in the limit $M \gg m$ and discuss.

Problem 2

Three particles of equal mass $m = m_1 = m_2 = m_3$ are constrained to slide around a frictionless circular track and are connected by three equal length springs that have spring constants k, k, and $2k$, respectively. See the below diagram. Initially, the system is in equilibrium and the springs are initially without any tension (i.e. they are neither stretched nor compressed).



(a) Write down the Lagrangian. Obtain the Equations of Motion (EOM) for this normal mode problem and express the EOM as an eigenvalue equation: $\Omega|\omega\rangle = \omega^2|\omega\rangle$, where Ω is a 3×3 matrix and ω is the oscillation frequency of a normal mode.

(b) Show by direct substitution into the eigenvalue equation in part (a) above, that there is a zero-frequency normal mode associated with an overall "rotation" of the system. Write the corresponding normalised eigenvector $|\omega_1\rangle$.

(c) Considering the symmetry of the configuration, write down another normal mode unit-vector $|\omega_2\rangle$ for the case when particle 1, opposite the spring with spring constant $2k$, remains in a fixed position. Using direct substitution into the eigenvalue equation of (a), check that $|\omega_2\rangle$ is indeed an eigenvector of Ω.

(d) Use $|\omega_1\rangle$ and $|\omega_2\rangle$ (or any other method you wish) to determine the final normal mode unit-vector $|\omega_3\rangle$. Via direct substitution into the eigenvalue equation of (a), show that $|\omega_3\rangle$ is also an eigenvector of Ω.

(e) Find the resulting time-dependent motion of the system if particle 1 is initially displaced from its equilibrium position by an amount x_0. You may leave your answer in terms of ω_1, ω_2, and ω_3, if you wish.

Problem 3

A ball of putty of mass m travels at speed v towards another ball of putty, also of mass m, which is at rest (but which is free to move) in the Lab frame of reference. They "collide" and "**stick**" together forming a new object.

(a) Assume that the collision is "head-on" and the speed v of the incoming ball is *non-relativistic*, i.e. much smaller than c the speed of light. Determine the final speed V and mass M of the new non-relativistic object after the collision.

(b) Again assume that the collision is "head-on", but now assume that the speed v of the incoming ball is *relativistic* such that it can not be neglected compared with c, the speed of light. Determine the final speed V and rest mass M of the new relativistic object after the collision.

(c) Compare the non-relativistic case (a) with the relativistic case (b) by filling in the below table with *"Yes"* or *"No"* answers.

	non-Relativistic case	Relativistic case
Is total kinetic energy conserved?		
Is total mass conserved?		

For any *"No"* answer you provide, explicitly show that the observable is not-conserved, and qualitatively explain why the observable is larger (or smaller) before versus after the collision.

(d) Reconsider the *non-relativistic* case, but now assume that there is an impact parameter r between the two balls of putty and that the two balls are instantaneously captured into a bound system by an infinitely stiff, massless string. Assume that the diameter of the balls are much smaller than r, so that they can be considered point-like particles – that is, assume that the new combined object after the collision consists of two point particles of mass m, separated by a fixed distance r. Determine the final speed and mass of the new combined object after the "collision." Is total kinetic energy conserved? Is mass conserved?

Problem 4

An idealised rigid body consists of three equal-mass points (of unit mass) that are placed in a force-free environment. The coordinates of the mass points are given by:

$$|r_1\rangle = \begin{pmatrix} 0 \\ \sqrt{22} \\ 0 \end{pmatrix} \quad |r_2\rangle = \begin{pmatrix} -\frac{3}{\sqrt{2}} \\ -\sqrt{\frac{11}{2}} \\ 0 \end{pmatrix} \quad |r_3\rangle = \begin{pmatrix} \frac{3}{\sqrt{2}} \\ -\sqrt{\frac{11}{2}} \\ 0 \end{pmatrix}$$

(a) Find the principle axes of the rigid body.

(b) In the CM frame of reference, find the moments of inertia, I_1, I_2, I_3, corresponding to the principle axes in part (a). Define the (normalised) principal axes $|I_1\rangle$, $|I_2\rangle$, and $|I_3\rangle$ in such a way that $I_1 < I_2 < I_3$.

(c) Show that a sufficient condition for a steady rotation of the rigid body is for the angular velocity to be along one of the principal axes of the body.

(d) Show that the steady rotation about the $|I_1\rangle$ axis is guarenteed to be stable against small perturbations. That is, write down the Equations of Motion for the rigid body and assume that the angular velocity about the $|I_1\rangle$ axis contains infinitesimal components along the other two principal axes, $|I_2\rangle$, and $|I_3\rangle$; then work to leading order in those infinitesimal quantities.

Problem 5

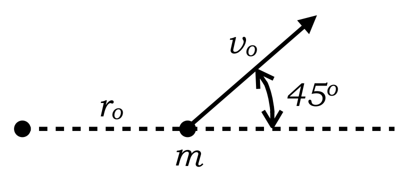
A particle of mass m moves under the influence of the central potential:

$$V(r) = -k/r^4$$

(a) Show that the motion occurs in a plane. Hence, use polar coordinates to write the Lagrangian for the system. Determine all constants of the motion.

(b) Make a plot of the effective potential for the radial motion of the particle. Give the general condition for a circular orbit. Does the above potential support circular orbits? If so, determine them in terms of constants of the motion; are they stable?

(c) At time $t = 0$ the particle is at $r = r_0$ and is moving with a velocity $v = v_0$ directed at an angle of 45^o with respect to the radial outward direction.



Write down the condition for the particle to escape to infinity. From these initial conditions, calculate the minimum value v_0 for which the particle is guarenteed to escape to infinity. You may work in and leave your answer in units for which $k = m = 1$.