1. (a) Let A be a linear operator on a Hilbert space $\mathcal{H}$. Assume that there exists an orthonormal basis for $\mathcal{H}$ consisting of eigenvectors of A. Show that $A = UB$, where U is a unitary operator on $\mathcal{H}$, and B is a Hermitian operator on $\mathcal{H}$ with no negative eigenvalues.

(b) Consider a quantum system associated with an N-dimensional Hilbert space $\mathcal{H}$. Let the Hamiltonian operator of the system be given by $H = A - B^2$, where A and B are Hermitian operators on $\mathcal{H}$. Show that the ground state energy E_0 of the system satisfies $E_0 \leq \frac{1}{N}\,\mathrm{Tr}(A)$. Find an explicit choice of A and B (with $H \neq 0$) such that the above inequality is saturated.

2. (a) Consider a spinless particle in three spatial dimensions moving under the influence of a spherically symmetric potential $V(r)$. The energy eigenvalues of the system are labelled as $E_{n,\ell}$, where $n = 0, 1, 2, \ldots$ is the radial quantum number, and $\ell = 0, 1, 2, \ldots$ is the total orbital angular momentum quantum number. Show that $E_{0,\ell} < E_{0,\ell+1}$ for any ℓ.

(b) Consider a spinless particle in one spatial dimension moving under the influence of the potential $V(x) = \lambda\,\delta(x + b) + \mu\,\delta(x - b)$, where λ, μ, and b are real numbers with $\lambda + \mu < 0$. Show that the system posseses at least one bound state. If $\lambda \cdot \mu < 0$, can there be more than one bound state?

3. (a) Consider a spinless particle of mass M in one spatial dimension moving under the influence of the potential $V(x) = \frac{1}{2}M\omega^2 x^2 + \lambda\,|x|$, where ω and λ are real numbers. (i) For "small" λ, calculate the ground state energy E_0 to first order in λ. (ii) In the limit as $\lambda \to +\infty$ and $\lambda \to -\infty$, find the energy splitting $E_1 - E_0$, where E_1 denotes the first excited state energy.

(b) Consider a quantum system associated with an infinite-dimensional Hilbert space $\mathcal{H}$. Let the Hamiltonian operator of the system be given by $H = H_0 + \lambda\,H_1$, where λ is a (dimensionless) real number, $H_1 = \mu\sum_{n,m=0}^{\infty}|\psi_n^{(0)}\rangle\langle\psi_m^{(0)}|$ (with μ real), and $H_0\,|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(0)}\rangle$ $(n = 0, 1, 2, \ldots)$ with $\langle\psi_n^{(0)}|\psi_m^{(0)}\rangle = \delta_{nm}$ and $\sum_{n=0}^{\infty}|\psi_n^{(0)}\rangle\langle\psi_n^{(0)}| = \mathrm{I}$. (The eigenvalues of H_0 are assumed non-degenerate.) Denote by E_s the s-th excited state energy level of H $(s = 0, 1, 2 \ldots)$, and for $|\lambda| \ll 1$ write $E_s = E_s^{(0)} + \lambda\,E_s^{(1)} + \lambda^2 E_s^{(2)} + \ldots$. For all s, calculate $E_s^{(1)}$. Assuming that $E_{n+2}^{(0)} - E_{n+1}^{(0)} = E_{n+1}^{(0)} - E_n^{(0)}$ for all n, show that $E_s^{(2)}$ diverges for all s.

4. Consider a machine capable of producing qubits in any one of the states $|\psi_\alpha\rangle = \frac{1}{\sqrt{2}}(\mathrm{e}^{i\pi\alpha}\,|1\rangle + \mathrm{e}^{-i\pi\alpha}\,|2\rangle)$, $0 \leq \alpha \leq 1$, where $\{|1\rangle, |2\rangle\}$ is an orthonormal basis for the associated two-dimensional Hilbert space $\mathcal{H}$. More specifically, when a button is pressed on the machine, it spits out a single qubit in a state chosen from among the $|\psi_\alpha\rangle$'s in such a way that over a large number of trials (that is, button pushes) the distribution of these qubit states is characterised by the classical probability density $p(\alpha) = 2\,\alpha$. Now suppose that an experiment is set up such that each time the button is pressed on the machine, an immediate measurement of the observable $A = \lambda\,(|1\rangle\langle 2| + |2\rangle\langle 1|)$ is made on the resulting qubit $(\lambda > 0)$.

(a) What are the possible results of such a measurement? After a large number of trials, what fraction of the total measurements yielded each of these possible results?

(b) Suppose that in one of the above measurements the lesser of the two possible outcomes is obtained. What is the state of the qubit immediately after this measurement? If the Hamiltonian operator of the system is given by $H = \mu\,|1\rangle\langle 1|$ (μ real), what is the state of the system at any later time?

5. (a) Consider a system of 3 spinless particles in three spatial dimensions. Particles 1 and 2 are identical, with common mass M, while particle 3 is "infinitely heavy" (with position fixed at the origin of coordinates). Let the potential for the 3-body system be a sum of 2-body contributions: $V(\vec{r}_1, \vec{r}_2) = V_{12}(\vec{r}_1, \vec{r}_2) + V_{13}(\vec{r}_1) + V_{23}(\vec{r}_2)$, where $\vec{r}_i$ is the position vector of particle i $(i = 1, 2)$. Further assume that $V_{i3}(\vec{r}_i) = \frac{1}{2}M\omega^2(\vec{r}_i)^2$ (where $i = 1, 2$ and ω is a real number) and $V_{12}(\vec{r}_1, \vec{r}_2) = \frac{2\,\hbar^2}{M(\vec{r}_1 - \vec{r}_2)^2}$. Determine the exact ground state energy and wavefunction of the system.

(b) Consider the set $W = \{L_1, L_2, L_3, S_1, S_2, S_3, J_1, J_2, J_3, L^2, S^2, J^2\}$, where L_i, S_i, and J_i $(i = 1, 2, 3)$ are the usual three-dimensional orbital, spin, and total angular momentum operators (for a single particle), respectively. Find all pairs of elements of W which can be simultaneously measured. Compute $[\,J^2,\, \vec{S}\cdot\vec{L}\,]$.