2013 UIC Physics Qualifying Exam, Electromagnetism Solution

1. A static electric field with spherical symmetry is described by $\boldsymbol{E} = (V_0/R)\exp(-r/R)\hat{\boldsymbol{r}}$.

   Determine the charge density $\rho(r)$.

   Find the total charge of the system.

   Find the static electric potential $V(r)$.

   Find the electrostatic energy by explicitly evaluating two different integrals, $\frac{\epsilon_0}{2}\int E^2 d^3\boldsymbol{r}$ and $\frac{1}{2}\int \rho V d^3\boldsymbol{r}$.

   A small test charge $+q$ is released at rest at the radial location $r = R\log 2$. What is the kinetic energy when it reaches a point far away?

The enclosed charge $Q(r)$ inside r is given by Gauss Law,

$$Q(r) = 4\pi r^2(\epsilon_0 V_0/R)\exp(-r/R) \ .$$

Thus the overall charge is $\boxed{Q(\infty) = 0}$ as expected because the total flux at infinity tends to zero.

$$dQ/dr = 4\pi(2r - r^2/R)(\epsilon_0 V_0/R)\exp(-r/R) = 4\pi r^2 \rho \ .$$

$$\text{So } \rho(r) = (2/r - 1/R)(\epsilon_0 V_0/R)\exp(-r/R) \ .$$

$$V(r) = \int_r^\infty E(r')dr' = (V_0/R)\int_r^\infty \exp(-r'/R)dr' = V_0\exp(-r/R) \ .$$

$$\int E^2 d^3\boldsymbol{r} = 4\pi\int_0^\infty r^2 e^{-2r/R}dr(V_0/R)^2 = \pi V_0^2 R \ .$$

$$\int \rho V d^3\boldsymbol{r} = (\epsilon_0 V_0^2/R)\int(2/r-1/R)e^{-2r/R}4\pi r^2 dr = (\epsilon_0 V_0^2 R)4\pi[2\times 1!/2^2 - 2!/2^3] = \epsilon_0\pi V_0^2 R \ .$$

Both integrals give the same electrostatic energy $\boxed{\frac{1}{2}\epsilon_0\pi V_0^2 R}$.

The initial potential energy of the test particle is $\boxed{qV_0/2}$, which becomes the final kinetic enegy at a very large r position.

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2. A circular circuit of radius a is folded into two perpendicular half circles. The center of the circuit is placed at the origin O. The fold-line is aligned with the y axis.

   The current I flows around the first half circle, which lies on the $x = z$ plane $(x > 0, z > 0)$, starting from $y = -a$ to $y = +a$. Then the current flows around the next half circle on the $x = -z$ plane $(x < 0, z > 0)$ from $y = +a$ back to the $y = -a$.

   Determine the magnetic field at the origin.

   Determine the *leading* dipole magnetic field $\boldsymbol{B}$ at a large distance $r \gg a$.

   A secondary circular loop is located at the spherical coordinates of fixed $r = R$, $\theta = 60°$ and varying ϕ. Find the leading dipole contribution of the mutual impedance between the two circuits (for $R \gg a$).

If the secondary loop carries another current I', determine the magnetic flux through the first small circuit due to I'.

For an unfolded circular circuit, $B. = \frac{\mu_0 I}{2a}$ by Biot Savart law. It is straightforward to show that for our case of the folder circle, $\boxed{\boldsymbol{B}(0) = \frac{\mu_0 I\sqrt{2}}{4a}\hat{\boldsymbol{z}}}$. The magnetic dipole is $\boldsymbol{m} = (\pi a^2 I/\sqrt{2})\hat{\boldsymbol{z}}$. So, for the far field,

$$\boldsymbol{B}(r,\theta) = \frac{\mu_0 \frac{\pi a^2 I}{\sqrt{2}}}{4\pi}\left(\frac{2\cos\theta}{r^3}\hat{\boldsymbol{r}} + \frac{\sin\theta}{r^3}\hat{\boldsymbol{\theta}}\right) .$$

Therefore, the flux through the secondary loop is

$$\Phi_{2\leftarrow 1} = 2\pi R^2 \int B_r(R) d(\cos\theta) = \frac{\mu_0 I a^2 \pi}{\sqrt{2}R}\int_{\frac{1}{2}}^{1} \cos\theta (d\cos\theta) = \frac{3\mu_0 I a^2\pi}{8\sqrt{2}R} , \quad M_{21} = \frac{3\mu_0 a^2 \pi}{8\sqrt{2}R} .$$

As $M_{21} = M_{12}$ reciprocally, the flux through the first circuit due to I' is

$$\Phi_{1\leftarrow 2} = \frac{3\mu_0 I' a^2 \pi}{8\sqrt{2}R} .$$

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3. An AC current $I(t) = I_0 \cos\omega t$ wraps around the inner long solenoid of radius a and returns around the outer long solenoid of radius b. The inner and outer solenoids, lying along the z-direction, have the same uniform winding density n, but in the opposite way of winding. The angular frquency ω is low enough that the quasi-static condition $\omega b/c \ll 1$ is satisfied. Find the induced electric field everywhere. Determine the Maxwell's displacement current density $\boldsymbol{J}_d$. Find the displacement current through a "transverse"area bounded by radii a and b and by a length ℓ in z.

Let the magnetic field $\boldsymbol{B}$ lie along the z axis (following the right hand rule). The Ampere's law gives

$$\boldsymbol{B} = \mu_0 n I(t) = \mu_0 n I_0 \cos\omega t \hat{\boldsymbol{z}} .$$

$$-\dot{\boldsymbol{B}} = \mu_0 n\omega I_0 \sin\omega t\hat{\boldsymbol{z}} , -\dot{\Phi} = \mu_0 n \omega I_0 (\sin\omega t)\pi(s^2 - a^2) .$$

However, the above expression applies only to the region between two solenoids. Otherwise, the magnetic field vanishes. The Faraday's law describes the non-electrostatic electric field,

$$\boldsymbol{E} = \frac{-\dot{\Phi}}{2\pi s}\hat{\boldsymbol{\phi}} = \tfrac{1}{2}\mu_0 n\omega I_0(\sin\omega t)(s - a^2/s)\hat{\boldsymbol{\phi}} , \text{ for } a < s < b ,$$

$$\boldsymbol{E} = \tfrac{1}{2}\mu_0 n\omega I_0(\sin\omega t)(b^2 - a^2)/s\hat{\boldsymbol{\phi}} , \text{ for } b < s ,$$

$$\boldsymbol{E} = 0 , \text{ for } s < a .$$

The displacement current density $\boldsymbol{J}_d$ is

$$\epsilon_0 \dot{\boldsymbol{E}} = \tfrac{1}{2}\epsilon_0\mu_0 n\omega^2 I(t)(s - a^2/s)\hat{\boldsymbol{\phi}} .$$

The current entering a "transverse" area bounded by radii a and b and by a length ℓ in z is given by

$$I_d = \tfrac{1}{2}\epsilon_0\mu_0 n\omega^2 I(t)\int_a^b (s - a^2/s)(ds\ell)$$
$$= (n\ell I(t))\,\frac{\omega^2}{2c^2}\left(\frac{b^2-a^2}{2} - a^2\log\frac{b}{a}\right)\ .$$

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4. An electromagnetic wave propagating in the free space is described by

$$\boldsymbol{E}(x,y,z,t) = (V_0/a)(\hat{\boldsymbol{z}})\cos(3x/a - 4y/a - \omega t)\ .$$

Your answers to the following questions must be in terms of V_0, a.

Find ω, the wavelength and the period of the wave. Determine the direction of propagation.

Explicitly give $\nabla\times\boldsymbol{E}$.

Determine the magnetic field of the wave.

Find the average electromagnetic energy density u.

Now this wave from the vacuum region $(3x - 4y < 0)$ approaches normally a non-magnetic media of the refractive index $n = \frac{7}{5}$ in the filled region $(3x - 4y > 0)$. Solve analytically the reflected electric field.

We can read from the expression $3x/a - 4y/a$ as $\boldsymbol{k}\cdot\boldsymbol{r}$ and conclude that $\boldsymbol{k} = (3/a)\hat{\boldsymbol{x}} - (4/a)\hat{\boldsymbol{y}}$. Its magnitude is $k = \sqrt{(\frac{3}{a})^2 + (\frac{4}{a})^2} = \frac{5}{a}$. The wavelength $\lambda = \frac{2\pi}{k} = \frac{2\pi a}{5}$. The period is $T = \frac{\lambda}{c} = \frac{2\pi a}{5c}$, $\omega = 5c/a$.

The wave propagates along the direction of the unit vector $\frac{3}{5}\hat{\boldsymbol{x}} - \frac{4}{5}\hat{\boldsymbol{y}}$.

$$\nabla\times\boldsymbol{E} = (V/a)\mathrm{Re}\ (i\boldsymbol{k}\times\hat{\boldsymbol{z}}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}) = (V/a)\mathrm{Re}\ (ie^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)})(-\tfrac{4}{a}\hat{\boldsymbol{x}} - \tfrac{3}{a}\hat{\boldsymbol{y}})\ .$$

$$\nabla\times\boldsymbol{E} = -(V_0/a)(-\tfrac{4}{a}\hat{\boldsymbol{x}} - \tfrac{3}{a}\hat{\boldsymbol{y}})\sin(3x/a - 4y/a - \omega t)$$
$$\omega\boldsymbol{B} = \boldsymbol{k}\times\boldsymbol{E}\ ,\quad \boldsymbol{B} = -\tfrac{V_0}{ac}(\tfrac{4}{5}\hat{\boldsymbol{x}} + \tfrac{3}{5}\hat{\boldsymbol{y}})\cos(3x/a - 4y/a - \omega t)$$
$$\langle u_{\mathrm{em}}\rangle = \tfrac{1}{2}\epsilon_0(V_0/a)^2\ .$$

As the interface is perpendicular to the incoming direction, we can redefine the usual coordinates such that the problem becomes one-dimensional as that isn textbooks. Let the electric amplitudes of incoming wave, the transmitted wave, and the reflected wave be denoted by E_i, E_t, E_r respectively. The Maxwell boundary condition implies

$$E_i + E_r = E_t\ ,\ E_i - E_r = nE_t\ ,\quad \text{so}\ \ E_r = -\frac{n-1}{n+1}E_i\ .$$

Using this result in our current setting, we obtain

$$\boldsymbol{E}_r(x,y,z,t) = -\tfrac{1}{6}(V_0/a)(\hat{\boldsymbol{z}})\cos(3x/a - 4y/a + 5ct/a)\ .$$

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5. Two overlapping charged lines A and B lie along the x axis of the frame O. Line A is static while Line B is moving at $\frac{3}{5}c$ in the $+x$ direction, as observed by O. Their line charge densities (i.e. charge per unit length) as measured by O are exactly opposite to each other, $\lambda(A) = \lambda_1 = -\lambda(B)$, thus overall neutral.

   Determine the electric $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ as functions of x, y, z in the frame O in terms of λ_1.

   Find the net force acting on a small stationary test charge $+q$ at a distance s away from the x axis.

   In the frame O' where Line B is static, what is the corresponding electric $\boldsymbol{E}'$ and magnetic field $\boldsymbol{B}'$ in terms of λ_1.

   The above test charge $+q$, static in O, turns out to be moving backward. What is the electric force and the magnetic force on it in the frame O'?

In the frame O, the system is neutral. $\boxed{\boldsymbol{E} = 0}$ as expected. There is a net current $\boxed{I = -\lambda_1 \frac{3}{5} c}$.

$$\boldsymbol{B} = -\frac{\mu_0 c \frac{3}{5} \lambda_1}{2\pi s} \hat{\boldsymbol{\phi}} \; .$$

Here s is the cylindrical radius $s = \sqrt{x^2 + y^2}$.

The net force on the static test charge is zero.

In the frame O', the two line charge densities are not equal in magnitude,

$$\lambda'(A) = \frac{\lambda_1}{\sqrt{1 - (\frac{3}{5})^2}} = \frac{5}{4}\lambda_1 \; ,$$

$$\lambda'(B) = -\lambda_1 \sqrt{1 - (\tfrac{3}{5})^2} = -\frac{4}{5}\lambda_1 \; ,$$

The net line charge density is $\lambda' = \frac{9}{20}\lambda_1$. The net current is $I' = \lambda'(A)(-\frac{3}{5}c) = -\frac{3}{4}\lambda_1 c$.

$$\boldsymbol{B}' = -\frac{\mu_0 c \frac{3}{4} \lambda_1}{2\pi s} \hat{\boldsymbol{\phi}} \; ,$$

$$\boldsymbol{E}' = +\frac{\mu_0 c^2 \frac{9}{20} \lambda_1}{2\pi s} \hat{\boldsymbol{s}} \; .$$

Now the test charge q moves backward with a speed $\frac{3}{5}c$. The electric force is

$$q\boldsymbol{E}' = +\frac{\mu_0 c^2 \frac{9}{20} \lambda_1 q}{2\pi s} \hat{\boldsymbol{s}} \; .$$

The magnetic force is

$$q(-\tfrac{3}{5}c)\hat{\boldsymbol{x}} \times \boldsymbol{B}' = -\frac{\mu_0 c^2 \frac{9}{20} \lambda_1 q}{2\pi s} \hat{\boldsymbol{s}} \; .$$

As expected, the net force is zero too.