University of Illinois at Chicago
Department of Physics

Electromagnetism

PhD Qualifying Examination

January 9, 2013
9.00 am 12:00 pm

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted toward the exams total score.

2013 UIC Physics Qualifying Exam, Electromagnetism Solution

1. A static electric field with spherical symmetry is described by $\boldsymbol{E} = (V_0/R)\exp(-r/R)\hat{\boldsymbol{r}}$.

   Determine the charge density $\rho(r)$.

   Find the total charge of the system.

   Find the static electric potential $V(r)$.

   Find the electrostatic energy by explicitly evaluating two different integrals, $\frac{\epsilon_0}{2}\int E^2 d^3\boldsymbol{r}$ and $\frac{1}{2}\int \rho V d^3\boldsymbol{r}$.

   A small test charge $+q$ is released at rest at the radial location $r = R\log 2$. What is the kinetic energy when it reaches a point far away?

2. A circular circuit of radius a is folded into two perpendicular half circles. The center of the circuit is placed at the origin O. The fold-line is aligned with the y axis.

   The current I flows around the first half circle, which lies on the $x = z$ plane $(x > 0, z > 0)$, starting from $y = -a$ to $y = +a$. Then the current flows around the next half circle on the $x = -z$ plane $(x < 0, z > 0)$ from $y = +a$ back to the $y = -a$.

   Determine the magnetic field at the origin.

   Determine the *leading* dipole magnetic field $\boldsymbol{B}$ at a large distance $r \gg a$.

   A secondary circular loop is located at the spherical coordinates of fixed $r = R$, $\theta = 60°$ and varying ϕ. Find the leading dipole contribution of the mutual impedance between the two circuits (for $R \gg a$).

   If the secondary loop carries another current I', determine the magnetic flux through the first small circuit due to I'.

3. An AC current $I(t) = I_0 \cos\omega t$ wraps around the inner long solenoid of radius a and returns around the outer long solenoid of radius b. The inner and outer solenoids, lying along the z-direction, have the same uniform winding density n, but in the opposite way of winding. The angular frquency ω is low enough that the quasi-static condition $\omega b/c \ll 1$ is satisfied. Find the induced electric field everywhere. Determine the Maxwell's displacement current density $\boldsymbol{J}_d$. Find the displacement current through a "transverse"area bounded by radii a and b and by a length ℓ in z.

4. An electromagnetic wave propagating in the free space is described by

$$\boldsymbol{E}(x, y, z, t) = (V_0/a)(\hat{\boldsymbol{z}})\cos(3x/a - 4y/a - \omega t) \ .$$

   Your answers to the following questions must be in terms of V_0, a.

   Find ω, the wavelength and the period of the wave. Determine the direction of propagation.

   Explicitly give $\nabla \times \boldsymbol{E}$.

   Determine the magnetic field of the wave.

Find the average electromagnetic energy density u.

Now this wave from the vacuum region $(3x - 4y < 0)$ approaches normally a non-magnetic media of the refractive index $n = \frac{7}{5}$ in the filled region $(3x - 4y > 0)$. Solve analytically the reflected electric field.

5. Two overlapping charged lines A and B lie along the x axis of the frame O. Line A is static while Line B is moving at $\frac{3}{5}c$ in the $+x$ direction, as observed by O. Their line charge densities (i.e. charge per unit length) as measured by O are exactly opposite to each other, $\lambda(A) = \lambda_1 = -\lambda(B)$, thus overall neutral.

   Determine the electric $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ as functions of x, y, z in the frame O in terms of λ_1.

   Find the net force acting on a small stationary test charge $+q$ at a distance s away from the x axis.

   In the frame O' where Line B is static, what is the corresponding electric $\boldsymbol{E}'$ and magnetic field $\boldsymbol{B}'$ in terms of λ_1.

   The above test charge $+q$, static in O, turns out to be moving backward. What is the electric force and the magnetic force on it in the frame O'?

Formulas of Electromagnetism

$$V_{\text{point charge}} = \frac{q}{4\pi\epsilon_0 r} = k_E \frac{q}{r} \ , \ \boldsymbol{E}_{\text{point charge}} = \frac{q\hat{\boldsymbol{r}}}{4\pi\epsilon_0 r^2} \ , \quad \boldsymbol{E} = -\boldsymbol{\nabla} V$$

$$V_{\text{dip}} = \frac{\hat{\boldsymbol{r}} \cdot \boldsymbol{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} \ , \ \boldsymbol{E}_{\text{dip}} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \ .$$

$$\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\boldsymbol{\ell}'}{|\boldsymbol{r}-\boldsymbol{r}'|^2} \times \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|} \ , \quad \boldsymbol{A} = \frac{\mu_0 I}{4\pi} \oint \frac{d\boldsymbol{\ell}'}{|\boldsymbol{r}-\boldsymbol{r}'|} \ , \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

$$\boldsymbol{J} d\tau' \leftrightarrow \boldsymbol{K} da' \leftrightarrow I d\boldsymbol{\ell}'$$

$$\boldsymbol{J} = \rho\boldsymbol{v} \ , \ \boldsymbol{K} = \sigma\boldsymbol{v} \ , \ I = dQ/dt \ .$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \ , \ \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \ , \ \oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}} \ .$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = 0 \ , \ \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\epsilon_0 \ , \ \oint \boldsymbol{E} \cdot d\boldsymbol{a} = Q_{\text{enc}}/\epsilon_0 \ .$$

$$\boldsymbol{A}_{\text{dip}} = \frac{\mu_0 \boldsymbol{m} \times \hat{\boldsymbol{r}}}{4\pi r^2} = \frac{\mu_0 m \sin\theta}{4\pi r^2}\hat{\boldsymbol{\phi}} \ , \ \boldsymbol{B}_{\text{dip}} = \frac{\mu_0 \ m}{4\pi r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \ .$$

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \longrightarrow \epsilon \boldsymbol{E} \ , \quad \boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) \longrightarrow \mu \boldsymbol{H}$$

$$\oint \boldsymbol{D} \cdot d\boldsymbol{a} = Q_{f,\text{enc}} \ , \quad \oint \boldsymbol{H} \cdot d\boldsymbol{\ell} = I_{f,\text{enc}} \ .$$

$$\nabla \cdot \boldsymbol{D} = \rho_f \ , \ \nabla \cdot \boldsymbol{P} = -\rho_b \ , \ \boldsymbol{\sigma}_b = \boldsymbol{P} \cdot \boldsymbol{n} \ .$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f \ , \ \nabla \times \boldsymbol{M} = \boldsymbol{J}_b \ , \ \boldsymbol{K}_b = \boldsymbol{M} \times \boldsymbol{n} \ .$$

$$W = \tfrac{1}{2} L I^2 \ , W = \tfrac{1}{2} I \Phi \ , \ W_{\text{mag}} = \frac{1}{2\mu_0} \int B^2 d\tau \ , \ V = IR \ , \ \mathcal{E} = -d\Phi/dt \ .$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\tfrac{\partial}{\partial t}\boldsymbol{B} \ , \quad \boldsymbol{\nabla} \times B = \mu_0(\boldsymbol{J} + \epsilon_0 \dot{\boldsymbol{E}}) \ , \quad I_{\text{disp}} = \epsilon_0 \tfrac{d}{dt} \int d\boldsymbol{a} \cdot \boldsymbol{E} \ .$$

$$\boldsymbol{f} = \rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} \ , \qquad .$$

$$u_{\text{em}} = \tfrac{1}{2}(\epsilon_0 E^2 + \tfrac{1}{\mu_0} B^2) \ , \quad \boldsymbol{S} = \tfrac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B} \ , \ \omega = vk \ , \ v = c/n \ .$$

Lorentz Transformation:

$$x' = \gamma(x - vt) \ , \ y' = y \ , \ z' = z \ , \ t' = \gamma(t - xv/c^2) \ , \ \gamma = 1/\sqrt{1 - v^2/c^2} \ , x^0 = ct \ .$$

$$c^2 = a^2 + b^2 - 2ab\cos C \ \text{(Cosine Law)}, \ \ e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3!}x^3 + \cdots \ .$$

$$(1+x)^n = 1 + nx + \tfrac{1}{2}n(n-1)x^2 + \tfrac{1}{6}n(n-1)(n-2)x^3 + \cdots \ .$$

$$d\tau = (dr)(r\sin\theta d\phi)(r d\theta) = r^2 \sin\theta d\phi d\theta dr \ ,$$

$$V_{\text{sphere}} = 4\pi R^3/3 \ , A_{\text{sphere}} = 4\pi R^2 \ , \ A_{\text{circle}} = \pi R^2 \ . \quad \sin(30^\circ) = \tfrac{1}{2} \ .$$