UNIVERSITY OF ILLINOIS AT CHICAGO
DEPARTMENT OF PHYSICS

Classical Mechanics
Ph.D. Qualifying Examination

8 January, 2013
9:00 to 12:00

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted toward the exams total score.

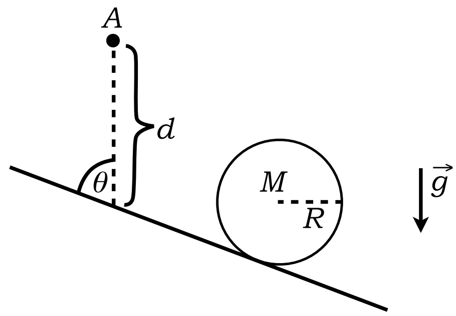
If needed, you may use the following table of integrals:

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left[\frac{x}{a}\right]$$

$$\int \frac{dx}{a^2 + b^2 \sin^2 x} = \frac{1}{a\sqrt{a^2 + b^2}} \tan^{-1}\left[\frac{\sqrt{a^2 + b^2}\tan x}{a}\right]$$

Problem 1

A homogenenous disk of radius R and mass M rolls without slipping on an inclined surface that makes an angle θ with respect to the vertical. The disk is constrained to be in contact with the inclined plane at all times. The disk is attracted to a point A located at a vertical distance d above the surface.



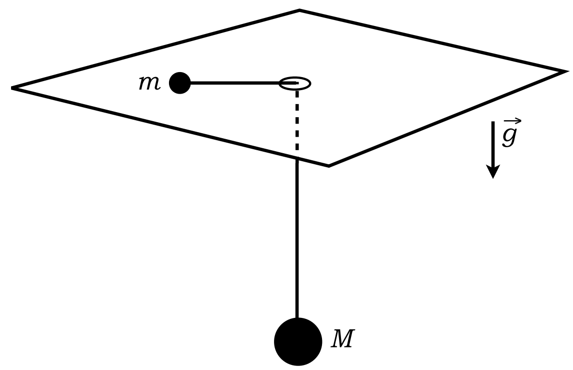
Assume that the force of attraction is proportional to the distance from the disk's center of mass to the force at point A; i.e. assume that $F = -kr$, where r is the distance from the point A to the disk's center of mass.

(a) Determine the equilibrium position of the disk, with respect to the position on the surface directly under point A (as shown in the figure above).

(b) Find the frequency of oscillations around the position of equilibrium.

Problem 2

A point mass m is moving on a frictionless horizontal table and is attached to a mass M by a massless, non-extendable string of total length l, which passes through a hole in the table. The mass M can only move vertically.



(a) (1) Identify appropriate generalised coordinates and write down the Lagrangian for the system. (2) Find and interpret the conserved quantity, whose coordinate is cyclic. (3) Find the radius r_0 of the stable circular orbit in terms of that conserved quantity. (4) What is the total energy of the circular orbit? Express your answer in terms of M, g, r_0, and l only.

(b) Assume an initial configuration such that exactly half the string lies on the table and the other half below the table, and the point mass m (on the table) is given an initial tangential speed of $\sqrt{lg}$. To make the calculations simpler, assume in this part that $m = M$. (1) Calculate the total energy of the system. (2) Plot the effective potential $V_{\text{eff}}(r)$. Describe *qualitatively* the orbit of the point mass on the table. (3) Calculate the maximum distance from the hole that this point mass reaches during its motion in terms of l.

Problem 3

A point mass m is fired vertically upwards with a speed v_0 from a point on the surface of the earth at a latitude λ. The angular velocity of the earth's rotation is $\vec{\omega}$, and the vertical axis $\hat{z}$ points in the effective vertical direction, such that the effective gravitational acceleration, $-g\hat{z}$ includes the centrifugal force due to the earth's rotation. Choose $\hat{y}$ east and $\hat{x}$ south. Ignore atmospheric effects.

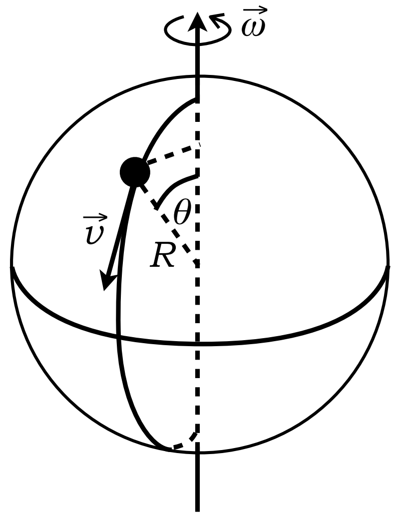
(a) By explicitely deriving the Coriolis effect to first order in ω, show that the total effective force on the point mass m is

$$\vec{F}_{\text{eff}} = -mg\hat{z} - 2m\omega(\cos\lambda)(v_0 - gt)\hat{y}$$

(b) Calculate the location of the point mass m after it lands? Check the sign of your answer by considering conservation of angular momentum.

Problem 4

A solid sphere of mass M and radius R rotates freely in space with an angular velocity ω about a fixed diameter. A point particle of mass m, initially at one pole, is constrained to move with a constant speed v on the surface of the sphere and to proceed along a line of longitude (i.e. a great circle).



(a) When the particle has reached the other pole, the rotation of the sphere will have been retarded. Why?

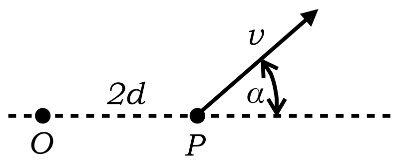
(b) Show that the angle by which the sphere is retarded due to the motion of the particle traveling from one pole to the opposite pole is

$$\Delta\alpha = \omega_0 T\Big(1 - \sqrt{\frac{2M}{2M+5m}}\Big)$$

where T is the total time required for the particle to move from one pole to the other and ω_0 is the initial angular velocity of the sphere before the particle begins to move.

Problem 5

A particle of mass m is moving under the attraction of an inverse square force of magnitude k/r^2. The particle was initially projected with speed $v = \sqrt{k/(2md)}$ from a point P a distance $2d$ from the force center O in a direction making an angle $\alpha = \pi/3$ with the line OP. (see figure below)



(a) Determine the energy of the particle, assuming that the potential energy of the particle at $r = \infty$ is zero.

(b) Determine the angular momentum of the particle.

(c) Determine the minimum and maximum distances of the particle from the force center in the subsequent motion.

(d) Determine the period of the motion.