University of Illinois at Chicago
Department of Physics

Thermodynamics and Statistical Physics
Qualifying Exam

January 6, 2012
9:00am-12:00pm

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted towards the exam's total score.

Mathematical Formulae

Notation:

$$\beta = \frac{1}{k_B T}$$

$$\mathrm{erf}\,(\mathrm{z}) = \frac{2}{\sqrt{\pi}} \int_0^{\mathrm{z}} \mathrm{dx} \exp\left(-\mathrm{x}^2\right)$$ erf is known as the error function

$$\mathrm{erfc}\,(\mathrm{z}) = \frac{2}{\sqrt{\pi}} \int_{\mathrm{z}}^{\infty} \mathrm{dx} \exp\left(-\mathrm{x}^2\right)$$ erfc is known as the complimentary error function

Integrals:

$$\int dx \ln x = x \ln x - x$$

$$\int \frac{dx}{x} = \ln x$$

$$\int_b^{\infty} dx\ \exp\left(-ax^2\right) = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,\mathrm{erfc}\left(\sqrt{\mathrm{a}}\mathrm{b}\right)$$

$$\int_0^a dx\ \mathrm{erfc}\,(\mathrm{x}) = \frac{1 - \exp\left(-\mathrm{a}^2\right)}{\sqrt{\pi}} + \mathrm{a}\,\mathrm{erfc}(\mathrm{a})$$

$$\int_0^n dx\ x^{1/2} \exp(-x) = \frac{\sqrt{\pi}}{2}\mathrm{erf}\left(\sqrt{n}\right) - \sqrt{n}\exp\left(-n\right)$$

$$\int_0^n dx\ x^{3/2} \exp(-x) = \frac{3}{4}\mathrm{erf}(\sqrt{\mathrm{n}}) - \frac{1}{2}\sqrt{\mathrm{n}}\exp(-\mathrm{n})(3 + 2\mathrm{n})$$

Expansions:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \qquad \text{for} \quad x < 1$$

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\mathrm{erfc}(\mathrm{x}) = \exp\left(-\mathrm{x}^2\right)\left[\frac{1}{\sqrt{\pi}\mathrm{x}} + \ \ldots\right] \qquad \text{for x} \to \infty$$

$$\sinh(x) = x + \ \ldots \qquad \text{for } x \to 0$$

$$\cosh(x) = 1 + \ \ldots \qquad \text{for } x \to 0$$

1. Consider a system consisting of N non-interacting particles each with isospin $I = 3/2$. The energies of the states with different I_z are given by

$$\begin{aligned} E(I_z = -3/2) &= E_1; \qquad & E(I_z = -1/2) &= E_2 \\ E(I_z = 1/2) &= E_3; \qquad & E(I_z = 3/2) &= E_3 \end{aligned}$$

with $E_1 < E_2 < E_3$ and $\Delta_{12} = E_2 - E_1 \ll \Delta_{23} = E_3 - E_2$.

a) Without using the partition function, give the value of the total energy, $\langle E \rangle$, at temperatures $T = 0$, $\Delta_{12} \ll T \ll \Delta_{23}$, and $\Delta_{23} \ll T$. Provide a justification for your results. Sketch $\langle E \rangle$ as a function of temperature.

b) What is the occupation of the I_z-states for temperature $T \to \infty$ Without using the partition function, give a value of the specific heat for temperature $T \to \infty$. Provide a justification for your results.

c) Without using the partition function, give the value of the average isospin per particle, $\langle I_z \rangle$, at temperatures $T = 0$, $\Delta_{12} \ll T \ll \Delta_{23}$, and $\Delta_{23} \ll T$. Provide a justification for your results. Sketch $\langle I_z \rangle$ as a function of temperature.

d) Using the partition function, compute the average isospin per particle, $\langle I_z \rangle$, in the limit $T \to \infty$. How does you result related to those in part c)?

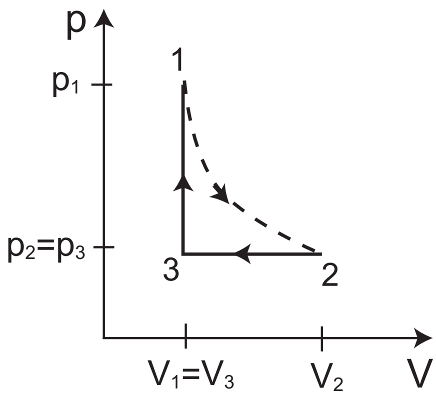
2. Consider an ideal gas of N_0 non-interacting spin-less particles each with kinetic energy

$$\varepsilon = \frac{m}{2} \vec{v}^{\,2}$$

that is contained in a box. The temperature of the gas is T_0, and the particles are uniformly distributed throughout the box.

Compute the total energy $E_0 = \langle E \rangle$ of the N_0 particles in the box. Next, one instantaneously removes all particles from the gas that possess a kinetic energy larger than nk_BT (n is an arbitrary real, positive number). How many particles remain in terms of N_0? What is the new total energy, E_{new} in terms of E_0? After the remaining particles have returned to equilibrium, what is the new temperature, T_{new} of the gas in terms of T_0?

3. Suppose one mole of an ideal gas is subjected to the cyclic process shown below (with temperature V_1, V_2 and V_3 in states $1, 2$ and 3, respectively)



$1 \Rightarrow 2$ is an isothermal expansion.

$2 \Rightarrow 3$ is an adiabatic expansion.

$3 \Rightarrow 1$ is an isochoric heating step.

All steps are reversible

a) What is the change in internal energy, ΔU, for the entire cyclic process $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

b) Use the First Law of Thermodynamics to calculate ΔU, δQ, and δW for the process $1 \Rightarrow 2$.

c) Use the First Law of Thermodynamics to calculate ΔU, δQ, and δW for the process $2 \Rightarrow 3$.

d) Use the First Law of Thermodynamics to calculate ΔU, δQ, and δW for the process $3 \Rightarrow 1$.

e) Is the total work done in a cycle positive or negative? What is the efficiency, η, of this cycle? In which limit does one obtain $\eta = 1$.

4. Consider a system consisting of M non-interacting molecules at temperature T. Each of these molecules possesses vibrations with energies

$$E_n = \hbar \omega_0 \left(n + \frac{1}{2} \right) \qquad \text{where} \quad n = 0, 1, 2, 3, ..., N_0$$

Let us first consider the case $N_0 = \infty$

a) Using the partition function, compute the total energy, $\langle E \rangle$, of the system for temperature $T \to 0$ and $T \to \infty$. Explain your results. At what temperature occurs the crossover from the $T \to 0$ to the $T \to \infty$ behavior of $\langle E \rangle$?

b) Compute $\langle n \rangle$ for $T \to \infty$. What is the physical interpretation of $\langle n \rangle$? What is the relation of $\langle n \rangle$ to the partition function and to $\langle E \rangle$?

c) Consider next the case where N_0 is a finite, integer number (i.e., $N_0 < \infty$). What is now the form of $\langle E \rangle$ for temperature $T \to \infty$

d) Compute the specific heat, C_V, of the system in the limit $T \to \infty$ for the two cases $N_0 = \infty$ and $N_0 < \infty$. Explain the difference in C_V between these two cases.

5. Consider a monoatomic ideal gas.

   a) Compute the entropy of an ideal gas as a function of T and V for constant particle number N starting from

$$dU = TdS - pdV$$

   b) Compute the chemical potential of the ideal gas as a function of p and T starting from the Gibbs-Duhem relation

$$SdT - Vdp + Nd\mu = 0$$