Quantum Solutions

1. (a) Since at $t = 0$ the system is in an eigenstate of the parity operator, we know that the associated wavefunction is either symmetric or anti-symmetric at this time. Moreover, since the potential is symmetric we have that the Hamiltonian operator commutes with $\hat{\Pi}$. Hence, the symmetry of the state will remain the same as the system evolves in time. Now the operator A is anti-symmetric (that is, $\hat{\Pi}\, A\, \hat{\Pi} = -A$). Therefore if $|\psi_\pm\rangle$ is an eigenstate of $\hat{\Pi}$ with eigenvalue ± 1, then $A\,|\psi_\pm\rangle$ is an eigenstate of $\hat{\Pi}$ with eigenvalue ∓ 1. Thus $\langle A\rangle_{|\psi_\pm\rangle} = \langle\psi_\pm|A|\psi_\pm\rangle = 0$, from which our result follows.

(b) Since $\langle\hat{\Pi}\rangle_{|\psi\rangle} = 0$, we have that $|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle + |\psi_-\rangle)$, where $|\psi_\pm\rangle$ is an eigenstate of $\hat{\Pi}$ with eigenvalue ± 1. In order to obtain the lowest value of $\langle H\rangle_{|\psi\rangle}$ in such a state, simple choose $|\psi_+\rangle$ to be the ground state (which is the symmetric state with the lowest energy, namely $E_0 = \frac{1}{2}\hbar\omega$) and $|\psi_-\rangle$ to be the first excited state (which is the anti-symmetric state with the lowest energy, namely $E_1 = \frac{3}{2}\hbar\omega$). A simple calculation then yields $\langle H\rangle_{|\psi\rangle} = (E_0 + E_1)/2 = \hbar\omega$.

2. (a) Since A and B are Hermitian, we have that $(A+B)^2$ is also Hermitian. But $2\,AB$ is Hermitian if and only if A commutes with B. Therefore we have that $(A+B)^2 = A^2 + B^2 + 2\,AB = 2\,AB$ (where the first equality uses $[A, B] = \hat{0}$), and hence $A^2 + B^2 = \hat{0}$. Taking the expectation of both sides of this last equation in an arbitrary state $|\psi\rangle$ yields $\langle A\psi|A\psi\rangle + \langle B\psi|B\psi\rangle = 0$, where we have used the Hermiticity of A and B. Since both terms on the LHS of this equation are ≥ 0 for any A and $|\psi\rangle$, we see that $\langle A\psi|A\psi\rangle = \langle B\psi|B\psi\rangle = 0$. But $\langle\phi|\phi\rangle = 0$ if and only if the vector $|\phi\rangle$ is the zero vector $|0\rangle$. Hence $A\,|\psi\rangle = B\,|\psi\rangle = |0\rangle$ for all states $|\psi\rangle$, which shows that $A = B = \hat{0}$.

(b) Let $|\phi\rangle$ be a unit vector in $\mathcal{H}$ which is orthogonal to $|\psi\rangle$. (The vector $|\phi\rangle$ is unique up to an overall phase.) Then $\{|\psi\rangle, |\phi\rangle\}$ is an orthonormal basis for $\mathcal{H}$. We can now (for example) choose $A = \alpha\,(|\psi\rangle\langle\phi| + |\phi\rangle\langle\psi|)$ and $B = \beta\,(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|)$, where α and β are non-zero real numbers. The eigenvalues of A are $\pm\alpha$, and the state of the system after the first measurement is $|\chi_\pm\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle \pm |\phi\rangle)$ if the value $\pm\alpha$ is obtained (each occurs with probability 1/2). We want to show that if we now measure B (whose eigenvalues are $\pm\beta$), there will be a non-zero probability of obtaining the value $-\beta$, in which case the state of the system after the measurement will be $|\phi\rangle$ (which is orthogonal to $|\psi\rangle$). But it is immediate from the form of the states $|\chi_\pm\rangle$ that the probability of obtaining $-\beta$ in the measurement of B is 1/2, regardless of the outcome of the first measurement.

3. (a) For any Hamiltonian operator H and any state $|\psi\rangle$, the variational theorem tells us that $E_0 \leq \langle\psi|H|\psi\rangle$, where E_0 is the lowest energy eigenvalue of H. In our case, if we make the choice $|\psi\rangle = |\psi_0^{(0)}\rangle$ we immediately obtain $\langle\psi|H|\psi\rangle = E_0^{(0)} + \lambda E_0^{(1)}$ (since $H_0|\psi_0^{(0)}\rangle = E_0^{(0)}|\psi_0^{(0)}\rangle$ and $E_0^{(1)} = \langle\psi_0^{(0)}|H_1|\psi_0^{(0)}\rangle$).

(b) Plugging the expansions for $|\psi_n\rangle$ and E_n into the time-independent Schrödinger equation $H\,|\psi_n\rangle = E_n\,|\psi_n\rangle$ and grouping terms of order λ^m $(m \geq 1)$ yields

$$H_0|\psi_n^{(m)}\rangle + H_1|\psi_n^{(m-1)}\rangle = \sum_{k=0}^{m} E_n^{(k)}|\psi_n^{(m-k)}\rangle \ .$$

Taking the inner product of both sides with $|\psi_n^{(0)}\rangle$ (and using $\langle\psi_n^{(0)}\,|\,\psi_n^{(m)}\rangle = 0$ for all $m \geq 1$ and the Hermiticity of H_0) then gives the desired result.

4. (a) The energy levels of the particle in the two-dimensional box is just the sum of the energy levels of two one-dimensional square-well problems, one of width L and the other of width $L/(N+1)$. Thus we have

$$E_{n_x,n_y} = \frac{\hbar^2\pi^2}{2ML^2}(n_x^2 + (N+1)^2 n_y^2)$$

(where $n_x, n_y = 1, 2, \ldots$). The result then follows by a simple computation.

(b) For a state with total angular momentum quantum number ℓ, the radial wavefunction satisfies $R(r) \sim r^\ell$ as $r \to 0$. Hence we have $\ell = 3$. Moreover, if the state has magnetic quantum number m, the ϕ-dependence of its wavefunction if $e^{im\phi}$. Hence we have $m = 2$. Finally, by substituting $R(r)$ in the (time-independent) radial Schrödinger equation and solving for $V(r)$ at large r, we find that $V(r)$ goes to a constant C as $r \to \infty$. Hence, there will be a continuum in the spectrum of H for $E \geq C$.

5. (a) The ^{11}B atom has 5 protons, 6 neutrons, and 5 electrons, each of which is a fermion. Hence ^{11}B is made of an even number of fermions, and is therefore a boson.

(b) In the absence of Coulomb repulsion (and spin-dependent forces), the problem reduces to 5 independent Hydrogenic atoms, each with nuclear charge $Z = 5$, where we also have to use the Pauli Exclusion Principle in order to properly distribute the spin one-half electrons in the Hydrogenic levels. In particular, 2 electrons will be in the 1S state, 2 in the 2S state, and one in the 2P state. Hence, in our approximation, the ground state energy of the Boron atom will be $E_0 = -Z^2(13.6 \text{ eV})(1 + 1 + 1/4 + 1/4 + 1/4) = -935$ eV.

(c) The degeneracy of the ground state is just the degeneracy of the 2P state for an electron, which is 6 since the quantum number m_ℓ of the 2P electron can be 1,0, or -1 (since this is an $\ell = 1$ state) and the quantum number m_s of this electron can be $\pm 1/2$ (since the electron has $s = 1/2$).

(d) Each of the spatial 2P basis states $|\psi_{2,1,m_\ell}\rangle$ ($m_\ell = 0, \pm 1$) is not rotationally symmetric, and the same will be true of any linear combination of these linearly independent degenerate states.

(e) Each Coulomb repulsion term increases the potential energy of the system, and hence the ground state energy will increase (but decrease in magnitude).