University of Illinois at Chicago
Department of Physics

Quantum Mechanics
PhD Qualifying Examination

January 3, 2012
9.00 am – 12:00 pm

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted toward the exam's total score.

Quantum Problems

1. (a) Consider a particle moving in one spatial dimension under the influence of a symmetric potential $V(x)$. If at time $t = 0$ the system is in an eigenstate of the parity operator $\hat{\Pi}$, show that the expectation value of the observable $A = \hat{x}^2\,\hat{p} + \hat{p}\,\hat{x}^2$ vanishes for all $t \geq 0$.

(b) Consider a particle of mass M moving under the influence of the one-dimensional potential $V(x) = \frac{1}{2}M\omega^2 x^2$. Assume that the state $|\psi\rangle$ of the system is such that $\langle\hat{\Pi}\rangle_{|\psi\rangle} = 0$. What is the minimum possible value of $\langle H\rangle_{|\psi\rangle}$ for such a state (where H is the Hamiltonian operator of the system)?

2. (a) Consider two Hermitian operators A and B on a Hilbert space which satisfy $(A+B)^2 = 2\,AB$. Show that $A = B = \hat{0}$.

(b) Consider a quantum system with two-dimensional Hilbert space $\mathcal{H}$, and assume that the system is in the state $|\psi\rangle$. Find two Hermitian operators A and B on $\mathcal{H}$ such that if a measurement of A is made on the system, followed immediately by a measurement of B, the state of the system immediately after the second measurement has a non-zero probability of being orthogonal to $|\psi\rangle$. (Hint: Consider an orthonormal basis for $\mathcal{H}$ containing $|\psi\rangle$, and express A and B in this basis.)

3. Consider a quantum system with Hamiltonian operator $H = H_0 + \lambda H_1$, and let $|\psi_n^{(0)}\rangle$ be the n-th excited energy eigenstate of H_0 with corresponding (non-degenerate) eigenvalue $E_n^{(0)}$. Assuming that $\lambda \ll 1$, the corresponding energy eigenstate $|\psi_n\rangle$ and energy eigenvalue E_n of H can be expanded as $|\psi_n\rangle = \sum_{m=0}^{\infty} \lambda^m |\psi_n^{(m)}\rangle$ and $E_n = \sum_{m=0}^{\infty} \lambda^m E_n^{(m)}$.

(a) For $n = 0$ (the ground state), show that $E_0 \leq E_0^{(0)} + \lambda E_0^{(1)}$. (Hint: Use the variational method with an appropriate trial wavefunction.)

(b) Show that $E_n^{(m)} = \langle\psi_n^{(0)}\,|\,H_1\,|\psi_n^{(m-1)}\rangle$ (for $m \geq 1$), where (as is standard) we have chosen $\langle\psi_n^{(0)}\,|\,\psi_n^{(m)}\rangle = 0$ for all $m \geq 1$.

4. (a) Consider a particle of mass M confined to a two-dimensional rectangular box, one side having length L and the other side having length $L/(N+1)$, where N is a fixed positive integer. (Inside the box, the particle is free.) Show that the spacing between the first N energy levels of the system are the same as for a particle of mass M confined to a one-dimensional square well of width L.

(b) Consider a particle moving under the influence of a spherically symmetric potential $V(r)$ in three spatial dimensions. Suppose that the system has a bound state wavefunction of the form $\psi_{n,\ell,m}(\vec{r}) = R(r) f(\theta)\, e^{2i\phi}$, where $R(r) \sim r^3$ as $r \to 0$ and $R(r) \sim r^a\, e^{-br}$ ($b > 0$) for $r \to \infty$. What are the values of the total angular momentum quantum number ℓ and the magnetic quantum number m for this state? Show that the Hamiltonian operator of the system has a continuous portion to its spectrum.

5. The most abundant stable isotope of atomic Boron is ^{11}B, whose nucleus contains 5 protons and 6 neutrons. Is ^{11}B a boson or a fermion? If we treat the nucleus as an infinitely heavy point-like object, and ignore the Coulomb repulsion between electrons (as well as all spin-spin and spin-orbit interactions), what is the total electronic energy for the ground state of ^{11}B? What is the degree of degeneracy of this state? Is any one of these degenerate eigenstates rotationally symmetric (about the nucleus)? If the Coulomb repulsion between electrons is now taken into account, will the ground state energy increase or decrease? (Show all your reasoning.)