University of Illinois at Chicago
Department of Physics

Electromagnetism
PhD Qualifying Examination

January 5, 2012
9.00 am – 12:00 pm

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted toward the exam's total score.

Question 1

A hollow metal sphere of radius R and mass M floats on an insulating dielectric liquid of density ρ and relative dielectric constant ε_r. When the metal sphere has no charge on it, it floats on the dielectric liquid as shown in Figure 1(a); i.e., the bottom of the sphere is $R/2$ below the surface of the dielectric liquid.

a) Determine the relationship between M and ρ when the sphere is ***not*** charged.
b) The hollow metal sphere is now charged with a charge Q. Draw a diagram that shows all the charges and explain why the sphere sinks further into the dielectric liquid when it is charged.
c) Find the magnitude of the charge Q to which the sphere must be charged in order for it to be half submerged as shown in Figure 1(b). Express your answer in terms of ρ, R, ε_r , the vacuum permittivity ε_0 , the acceleration due to gravity g, and other numerical factors.

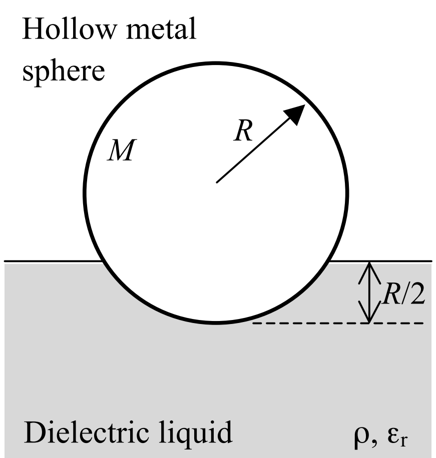


Figure 1(a)

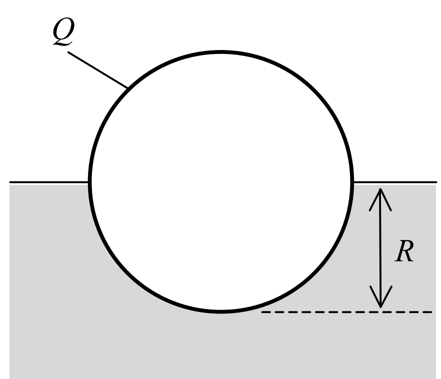
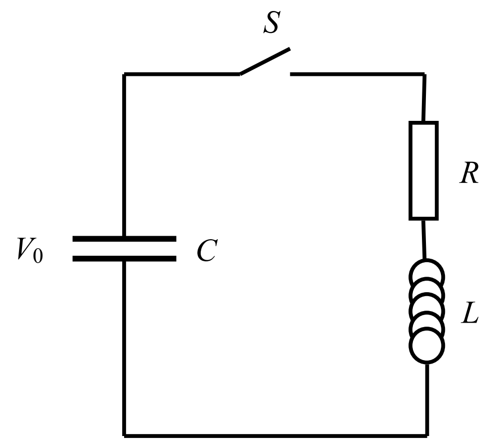


Figure 1(b)

Question 2

Consider the circuit shown below, in which for times $t < 0$ the capacitor of capacitance C is charged to a voltage V_0. At $t = 0$, the switch S is closed, allowing the capacitor to discharge through a resistor R and an inductor L placed in series.



a) Using Kirchoff's voltage law, write down the second-order differential equation describing the evolution of the charge q on the capacitor for times $t > 0$.

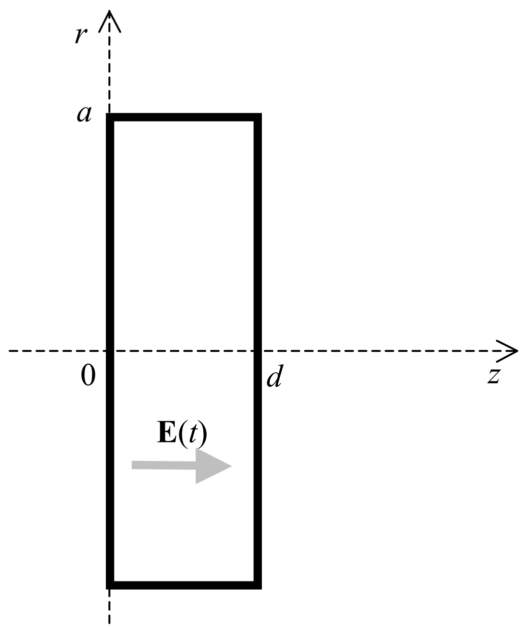
b) For times $t > 0$, solve the differential equation obtained in part (a) subject to the boundary conditions $q(t = 0) = q_0$ and $\left.\frac{dq}{dt}\right|_{t=0} = 0$.

c) Explain why the current in the circuit builds up to a maximum value and then decays to zero. Show that the time t at which the current in the circuit is a maximum is given by the relation

$$\tanh(\Omega t) = \frac{2\Omega}{\alpha},$$

where $\Omega = \sqrt{\omega^2 + \frac{\alpha^2}{4}}$ and $\alpha = \frac{R}{L}$ with $\omega = \frac{1}{\sqrt{LC}}$.

Question 3



A cylindrical 'pill-box' resonator of radius a and length d is driven at its fundamental TM_{010} mode for which the oscillating electric component of the RF field may be written as

$$\mathbf{E}(t) = \hat{\mathbf{z}} E_0 J_0\left(\frac{2.405r}{a}\right)\sin(\omega t + \phi)$$

where $J_0(x)$ is the Bessel function of zero order whose first zero is at $x = 2.405$.

a) What is the form of the magnetic component of the oscillating RF field in the cavity?
b) Verify that the average value of the Poynting vector (i.e., $\langle \mathbf{S} \rangle_{av.}$) is zero.
c) What is the stored energy of the oscillating TM_{010} mode?

Bessel function relations:

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+1)}\left(\tfrac{1}{2}x\right)^{2m+n} \quad \text{where } \Gamma(p) = (p-1)! \text{ for } p \text{ positive integer}$$

$$\frac{\partial J_n(x)}{\partial x} = J_{n+1}(x) \qquad \int_0^1 x\,dx\,J_n^2(\alpha x) = \tfrac{1}{2} J_{n+1}^2(\alpha)$$

Question 4

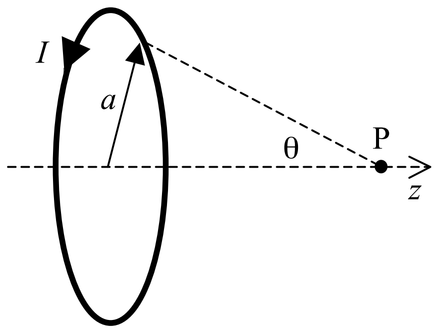


Figure 4(a)

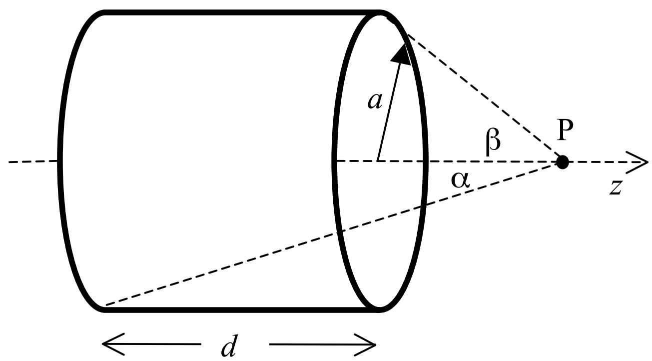


Figure 4(b)

a) Show that for a single wire loop of radius a carrying current I the axial magnetic field at point P in Figure 4(a) may be written as

$$\mathbf{B}(\theta) = \hat{\mathbf{z}}\frac{\mu_0 I}{2a}\sin^3\theta,$$

where θ is the angle subtended from the axis at point P to the circumference of the loop and μ_0 is the permeability of vacuum.

b) Use the result of part (a) to show that the axial magnetic field at point P for a solenoid of length d and radius a carrying current I in N turns (Figure 4(b)) is given by

$$B(\alpha,\beta) = \frac{\mu_0 NI}{2d}\left(\cos\alpha - \cos\beta\right),$$

where the front and back coils (i.e., ends) of the solenoid subtend angles α and β with its axis at point P. Verify that your answer reduces to the expected result for the field inside an infinitely long solenoid (i.e., $d >> a$).

c) Show that for small distances $z << d$ inside and close to the center of a narrow ($d >> a$) finite solenoid that the ***axial*** dependence of the magnetic field strength is parabolic in z and of the form

$$B(z) = \frac{\mu_0 NI}{d}\left[1 - \frac{2a^2}{d^2}\left(1 + \frac{12z^2}{d^2}\right)\right].$$

Question 5

A material can be anisotropic in either or both its refractive index and absorption. These optical properties are described by a permittivity tensor, $\underline{\underline{\varepsilon}}$, for the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ directions (i.e., a 3×3 matrix). The wave equation in a non-conducting, non-magnetic medium then reads

$$\nabla^2 \mathbf{E} - \mu_0 \underline{\underline{\varepsilon}} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad .$$

a) For a electromagnetic wave of frequency ω propagating in direction $\mathbf{k}$ with a polarization unit vector $\hat{\mathbf{e}}$ and amplitude $\mathbf{E}_0$ described by, $\mathbf{E}(\mathbf{r},t) = \frac{1}{2}\hat{\mathbf{e}}\mathbf{E}_0 \exp[i(\mathbf{k.r} - \omega t)] + c.c.$, show that the refractive index n experienced by the wave in the medium is given by the expression

$$n^2 = \frac{1}{\varepsilon_0}\left[\hat{\mathbf{e}}^* . (\underline{\underline{\varepsilon}}.\hat{\mathbf{e}})\right] \quad .$$

b) For $\hat{\mathbf{e}} = (\sin\theta, 0, \cos\theta)$ in the *x-z* plane, determine the refractive index experienced by the wave in a non-absorbing crystalline medium described by $\underline{\underline{\varepsilon}} = \varepsilon_0 \begin{pmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & (n_o + \Delta n)^2 \end{pmatrix}$, where n_o and $n_e = n_o + \Delta n$ are the ordinary and extra-ordinary refractive indexes of the unaxial crystal respectively.

c) What is the walk-off angle ϕ between the Poynting vector $\mathbf{S}$ and the wave vector $\mathbf{k}$ of the wave in the anisotropic medium if its magnetic field amplitude is given by $\mathbf{H} = (0, H_0, 0) = \hat{\mathbf{y}} H_0$?

d) What is the angle between $\mathbf{E}$ and $\mathbf{D}$ in the uniaxial crystal?