University of Illinois at Chicago
Department of Physics

Thermodynamics and Statistical Mechanics
Qualifying Examination

January 7, 2011
9:00 AM to 12:00 Noon

Full credit can be achieved from completely correct answers to <u>4 questions</u>. If the student attempts all 5 questions, all the answers will be graded, and the <u>top 4 scores</u> will be counted towards the exam's total score.

Equation Sheet

$$\int_{-\infty}^{\infty} \exp[-bx^2]\, dx = \sqrt{\frac{\pi}{b}}$$

$$\int_{0}^{\infty} x^2 \exp[-bx^2]\, dx = \frac{1}{4}\sqrt{\frac{\pi}{b^3}}$$

$$\int_{0}^{\infty} x^2 \exp[-x]\, dx = 2$$

$$\int_{0}^{\infty} dx \frac{x}{e^x - 1} = \frac{\pi^2}{6}$$

$$\int_{0}^{\infty} dx \frac{x}{e^x + 1} = \frac{\pi^2}{12}$$

$\int_{0}^{\infty} dx \frac{x^2}{e^x - 1} = 2\,\zeta(3)$, where $\zeta(3)$ can be considered to be just a number.

$$\int_{0}^{\infty} dx \frac{x^2}{e^x + 1} = \tfrac{3}{2}\,\zeta(3)$$

$$\bar{n} = \frac{1}{e^{(\varepsilon-\mu)/kT} \pm 1}$$

$$\sinh(x) = \tfrac{1}{2}(e^x - e^{-x})$$

$$\cosh(x) = \tfrac{1}{2}(e^x + e^{-x})$$

$$\frac{d}{dx}\left[\sinh(x)\right] = \cosh(x)$$

$$\frac{d}{dx}\left[\cosh(x)\right] = \sinh(x)$$

$$\sum_{m=0}^{\infty} x^m = \frac{1}{1-x}$$

$$\sum_{m=0}^{n} x^m = \frac{1-x^{n+1}}{1-x}$$

Problem 1

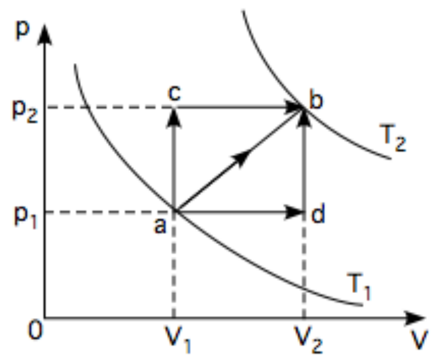
A gas of N identical classical non-interacting atoms is held in a potential $V(r) = ar$, where $r = \left(x^2 + y^2 + z^2\right)^{1/2}$. The gas is in thermal equilibrium at temperature T.

(a) Find the single particle partition function Z_1 of an atom in the gas. Express your answer in the form $Z_1 = AT^{\alpha}a^{-\eta}$ and provide an expression for the prefactor A and the exponents α and η. [Hint: convert the integral in r to spherical coordinates.]

(b) Find an expression for the entropy S of this classical gas.

Problem 2

A classical ideal gas is taken from state a to state b in the figure using three different paths: acb, adb, and ab. The pressure $p_2 = 2p_1$ and the volume $V_2 = 2\,V_1$.



(a) The heat capacity $C_V = \frac{5}{2}Nk$. Starting from the First Law of Thermodynamics derive a value for C_p. No credit will be given for this part if you just state the answer.

(b) Compute the heat supplied to the gas along each of the three paths, acb, adb, and ab, in terms of N, k, and T_1.

(c) What is the heat capacity C_{ab} of the gas for the process ab?

Problem 3

Consider a one-dimensional stretched elastic string that is fixed at its two ends and vibrates only in a direction perpendicular to its length. The string consists of a very large number N of atoms arranged in a single row. Let the energies of vibration be quantized in units of hf, where f is the vibration frequency. This string is in thermal equilibrium with a heat bath at temperature T.

(a) Determine an expression for the thermal energy of this string in terms of an integral over the variable $x = \varepsilon / kT$.

(b) Identify a characteristic temperature that separates low T and high T behavior. Determine an expression for the thermal energy of this string in the limit of low and high T. Comment on these results in the context of the equipartition theorem.

Problem 4

Consider a spherical drop of liquid water containing N_l molecules surrounded by N - N_l molecules of water vapor. The drop and its vapor may be out of equilibrium.

(a) Neglecting surface effects write an expression for the Gibbs free energy of this system if the chemical potential of liquid water in the drop is μ_l and the chemical potential of water in the vapor is μ_v. Rewrite N_l in terms of the (constant) volume per molecule in the liquid, v_l, and the radius r of the drop.

(b) The effect of the surface of the drop can be included by adding a piece $G_{\text{surface}} = \sigma A$ to the free energy, where σ is the surface tension ($\sigma > 0$) and A is the surface area of the drop. Write G_{total} with the surface piece expressed in terms of r. Make two qualitative hand-sketches of G_{total}: one sketch with $(\mu_l - \mu_v) > 0$ and one sketch with $(\mu_l - \mu_v) < 0$. Describe the behavior of the drop in these two cases.

(c) Under appropriate conditions, there is a critical radius, r_c, that separates drops which grow in size from those that shrink. Determine this critical radius.

(d) Assume that the vapor behaves as an ideal gas and recall that the chemical potential of an ideal gas is given by $\mu_v = \mu_v^o + kT \ln(p / p^o)$. Write the chemical potential difference $(\mu_v - \mu_l)$ in terms of the vapor pressure and a reference pressure p^o , where p^o is taken to be the pressure of a vapor in equilibrium with a large flat surface of water. Then, derive and comment on the dependence of the relative humidity p / p^o on r_c .

Problem 5

Consider a paramagnetic material whose magnetic particles have angular momentum J, which is a multiple of ½. The projections of the angular momentum along the z-axis can take $2J - 1$ values ($J_z = -J, -J+1, -J+2, ..., J$), which leads to $2J - 1$ allowed values of the z – component of a particle's magnetic moment ($\mu_z = -J\delta_\mu, -(J+1)\delta_\mu, ..., J\delta_\mu$). The energy of the magnetic moment in a magnetic field pointing in the $+z$ direction is $-\mu_z B$.

(a) Derive an expression for the partition function Z_1 of a single magnetic particle in a magnetic field B pointing in the $+z$ direction. Write your answer in terms of hyperbolic sin functions, where $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$. You may find it convenient to use the variable $b = \delta_\mu B\beta$, where $\beta = 1/kT$.

(b) Derive an expression for the average energy of the particle in part (a). Write your answer in terms of the hyperbolic cotangent function $\coth(x) = \frac{\cosh(x)}{\sinh(x)}$.

(c) Use the expression for the average energy in part (b) to determine the magnetization M (the average z-component of the total magnetic moment) of a system of N identical, independent magnetic particles. Comment on its behavior as $T \to 0$.