# University of Illinois at Chicago
# Department of Physics

# Quantum Mechanics
# Qualifying Examination

# January 3, 2011
# 9:00 am - 12:00 noon

Full credit can be achieved from completely correct answers to **4 questions**. If the student attempts all 5 questions, all of the answers will be graded, and the **top 4 scores** will be counted toward the exam's total score.

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Useful integration formulas are

$$\int_0^\infty x^n e^{-ax} dx = n!/a^{n+1} \ , \quad \text{valid for complex } a \text{ as long as Re}(a) > 0.$$

$$\int_{-\infty}^\infty e^{-ax^2} dx = \sqrt{\pi/a} \ , \quad \text{valid for complex } a \text{ as long as Re}(a) \geq 0.$$

$$\int_{-\infty}^\infty e^{-\gamma x^2} dx = \sqrt{\pi/\gamma} \ , \ \int_{-\infty}^\infty x^2 e^{-\gamma x^2} dx = \frac{1}{2\gamma}\sqrt{\pi/\gamma} \ , \ \int_{-\infty}^\infty x^4 e^{-\gamma x^2} dx = \frac{3}{4\gamma^2}\sqrt{\pi/\gamma} \ .$$

1. For a 1-dimensional simple harmonic quantum oscillator, $V(x) = \frac{1}{2}m\omega^2 x^2$, it is more convenient to describe the dynamics by dimensionless position parameter $\rho = x/a$ ($a = \sqrt{\frac{\hbar}{m\omega}}$) and dimensionless energy $\epsilon = E/(\frac{1}{2}\hbar\omega)$.

(i) Write down the time-independent Schrodinger equation in terms of $\rho$ derivatives on the eigenfunction $u(\rho)$. Show that it is equivalent to set $\omega = 2$, $2m = 1$, $\hbar = 1$.

Directly show that $u_0(\rho) \sim e^{-\frac{1}{2}\rho^2}$ satisfies your Schrodinger equation. Explain why it is the ground state and give its energy $\epsilon_0$ in the dimensionless unit.

Directly show that $u_1(\rho) \sim \rho e^{-\frac{1}{2}\rho^2}$ satisfies the Schrodinger equation as the first excited state. Also give its energy $\epsilon_1$ in the dimensionless unit.

The initial wave function $u(\rho, 0)$ is described by $u(\rho, 0) = N(2\sqrt{2}\rho - \frac{3}{2})\exp(-\frac{1}{2}\rho^2)$, with $N$ as the normalization constant.

(ii) Find the average energy.

(iii) The momentum operator in the reduced unit is $\mathtt{p} = -id/d\rho$. Find its mean value $\langle \mathtt{p} \rangle$ initially.

(iv) We use a dimensionless time parameter $\tau = \frac{1}{2}\omega t$ to study how the wave evolves. Write down the explicit $\tau$ dependence of the wave function.

(v) Find $\langle \mathtt{p} \rangle$ as a function of time $\tau$.

2. In the beginning, a non-relativistic particle of mass $m = \frac{1}{2}$ propagates freely as a wave packet with a mean wave number $q = \frac{4}{27}$. We choose a unit system such that $\hbar = 1$. Find the group velocity and the phase velocity.

Let this wave propagate from the remote left toward a repulsive potential $V(x) = \frac{1}{2}(1 - 9x^2)$ in the region $[-\frac{1}{3}, \frac{1}{3}]$. The potential vanishes otherwise. We can treat the potential as a delta-function potential $G\delta(x)$. Give quantitative reasons why. Figure out the equivalent strength $G$.

Derive the probabilities of transmission and reflection in the delta potential approximation, and give the numerical results for the given inputs.

3. For matrices $A, B, C$, show that $[AB, C] = A[B, C] + [A, C]B$. Let $S_1$ (or $S_x$) be the $x$-component of the spin vector operator, etc. Using the algebra of angular momentum, $[S_x, S_y] = i\hbar S_z$, and cyclic permutations for a general spin ($S = \frac{1}{2}, 1, \frac{3}{2}, \cdots$), simplify $[S_x^2, S_z]$ and $[S_y^2, S_z]$ in terms of $S_x S_y$ or $S_y S_x$. Then calculate $[\boldsymbol{S}^2, S_z]$. The trace of a matrix $A$ is defined as Tr $A = \sum_m \langle m|A|m\rangle$ summing over the basis vectors $m$. Show that Tr $(AB)$ = Tr $(BA)$ by working out the component sum. Based on some earlier steps, show that Tr $(S_x S_y) = 0$.

An unknown particle ($X$) of spin $S$ and mass $M_X$ couples to the fixed target nucleus of spin $I$ by a feeble spin-dependent contact interaction

$$\mathcal{V} = g\delta^3(\boldsymbol{r})\boldsymbol{S} \cdot \boldsymbol{I}$$

.

What are possible numbers of $\boldsymbol{S}\cdot\boldsymbol{I}$ in general? Show all these possibilities for the special case $S=\frac{1}{2}$, $I=\frac{9}{2}$.

Justify the following trace relations,

$$\text{Tr }(S_iS_j) = \; C_S\delta_{ij}\ , \qquad \text{or similarly} \qquad \text{Tr }(I_iI_j) = C_I\delta_{ij}\ ,$$

when the corresponding $m_S$, or $m_I$ states are summed respectively. Determine coefficients $C_S$ and $C_I$ in terms of general $S$ and $I$ respectively.

Calculate the transition probability

$$\sum |\langle \boldsymbol{k}k, m'_S, m'_I|\mathcal{V}|\boldsymbol{k}, m_S, m_I\rangle|^2$$

from an incoming $X$ plane wave described by $e^{i\mathbf{k}\cdot\mathbf{r}}$ to the outgoing $X$ wave $e^{i\mathbf{k}'\cdot\mathbf{r}}$. The sum adds up all spin states $m_S$ and $m'_S$ of the initial and final spin states of the $X$ particle, as well as $m_I$ and $m'_I$ of nucleus.

After averaging the initial spins of the $X$ particle and the nucleus, find the total cross section in the Born approximation. (Hints: The usual potential scattering in the Born approximation is

$$\frac{d\sigma}{d\Omega} = \frac{M^2}{4\pi^2\hbar^4}\left|\int d^3r V(r)e^{i(\mathbf{k}_f-\mathbf{k}_i)\cdot\mathbf{r}}\right|^2\ .$$

The above formula has to be generalized to incorporate the spins of $S$ and $I$.)

4. A quantum particle is in a two dimensional potential $V(r) = -V_0\exp(-r/a)$ (with $V_0>0$). Let the trial wave function be $\varphi(r;\beta) = Ce^{-\beta r}$. Determine the normalization $C$ in terms of the attenuation parameter $\beta$.

Find the average position and average momentum $\bar{x} = \langle x\rangle$, $\bar{y} = \langle y\rangle$, $\bar{p}_x = \langle p_x\rangle$, $\bar{p}_y = \langle p_y\rangle$.

Determine $\langle \boldsymbol{r}^2\rangle$ and $\langle \boldsymbol{p}^2\rangle$. (Hints: $\int d^2\mathbf{r}\psi^*(\mathbf{r})\nabla^2\psi(\mathbf{r}) = -\int d^2\mathbf{r}|\nabla\psi(\mathbf{r})|^2$.)

What are $\langle x^2\rangle$ and $\langle p_x^2\rangle$?

Calculate the product $\langle (x-\bar{x})^2\rangle\langle (p_x-\bar{p}_x)^2\rangle$, and simplify the result in comparison with the Heisenberg uncertainty.

Find the average kinetic energy and average potential energy.

Now we set $\frac{\hbar}{2m}=1$, $V_0=8$, $a=\frac{1}{2}$. Verify that $\beta=1$ minimizes the energy expectation value.. Using the variational principle, estimate the ground state energy.

5. The dynamic of a three-state system of configurations $|1\rangle$,$|2\rangle$,$|3\rangle$, is governed by the Hamiltonian $\mathcal{H}_0 = -\sum_{i,j}|i\rangle\langle j|$, which is

$$-|1\rangle\langle 1| - |1\rangle\langle 2| - |1\rangle\langle 3|$$

$$-|2\rangle\langle 1| - |2\rangle\langle 2| - |2\rangle\langle 3|$$

$$-|3\rangle\langle 1| - |3\rangle\langle 2| - |3\rangle\langle 3|\ .$$

Work out the Hamiltonian matrix elements $\langle i|\mathcal{H}_0|j\rangle$, and present the corresponding $3\times 3$ matrix.

Find eigen energies as well as the corresponding eigenstates.

A small perturbation $gV = g|1\rangle\langle 1|$ is applied ($g \ll 1$). Find the 1st order and 2nd order corrections of the ground state energy .