Solutions

1. Plum Pudding Model

(a) Find the corresponding electrostatic potential inside and outside the atom.

For $r \leq R$

$$\nabla^2 V_{in} = -\frac{\rho_0}{\varepsilon_0}.$$

The solution can be found by integrating twice,

$$V_{in} = -\frac{\rho_0}{6\varepsilon_0} r^2 + \frac{a}{r^2} + b,$$

where a and b are constants. The potential at the center of the atom has to be finite, so $a = 0$.

Finally,

$$V_{in} = -\frac{\rho_0}{6\varepsilon_0} r^2 + b.$$

For $r > R$

$$\nabla^2 V_{out} = 0.$$

The solution can be found by integrating twice,

$$V_{out} = -\frac{c}{r} + d.$$

But for large r, the potential has to go to zero, therefore $d = 0$.

Using the boundary condition that the potential has to be continuous at $r = R$:

$$\frac{\rho_0}{6\varepsilon_0} R^2 + b = \frac{c}{R}.$$

(b) Find the electrostatic vector-field inside and outside the atom.

For $r R$:

$$\vec{E}_{out} = -\vec{\nabla} V_{out} = \frac{c}{r^2}\hat{r}.$$

For $r \leq R$:

$$\vec{E}_{in} = -\vec{\nabla} V_{in} = \frac{\rho_0 \vec{r}}{3\varepsilon_0}.$$

Using the continuity of $\vec{E}$ at $r = R$:

$$\frac{c}{R^2} = \frac{\rho_0 R}{3\varepsilon_0} \Leftrightarrow c = \frac{\rho_0 R^3}{3\varepsilon_0}$$

In summary:

$$\vec{E}_{in} = \frac{\rho_0 \vec{r}}{3\varepsilon_0}, \vec{E}_{out} = \frac{\rho_0 R^3}{3\varepsilon_0}\frac{\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q_{total}}{r^2}\hat{r}, b = -\frac{\rho_0 R^2}{6\varepsilon_0}, \text{ and } V_{in} = -\frac{\rho_0}{6\varepsilon_0}\left(r^2 + R^2\right)$$

(c) Electrons inside the Plum Pudding Model atom will oscillate. Explain why!

The potential energy of an electron inside the atom is

$$U = eV(r) = -\frac{e\rho_0}{6\varepsilon_0}\left(r^2 + R^2\right) = Ar^2 + const$$

and the force on the electron is:

$$\vec{F} = q\vec{E}_{in} = -\frac{e\rho_0 \vec{R}}{3\varepsilon_0}$$

(d) Find the frequency of this oscillation.

Use

$$m\partial_t^2 r = -\frac{e\rho_0 r}{2\varepsilon_0} \Leftrightarrow \partial_t^2 r = -\frac{2\varepsilon_0}{e\rho_0 r}$$

Using

$$r = r_0 e^{i\omega t}$$

we can find

$$\omega = \sqrt{\frac{\rho_0 e}{3m\varepsilon_0}}$$

2. Dielectric Sphere

(a) Determine the boundary conditions for the given setup.

The boundary conditions can be summarized as follows

1. $\lim_{r\to\infty} V(\vec{r}) = -E_0 r\cos\theta$
2. $\lim_{r\to 0} V(\vec{r})$ remains finite.
3. $V(R)_{in} = V(R)_{out}$
4. $\varepsilon_{r,in}\partial_r V_{in} - \partial_r V_{out} = 0$

(b) Find the potential inside and outside the dielectric sphere. Explain your approach! Use the method of separation of variables in spherical coordinates to determine the potential inside and outside the sphere. In combination with the first two boundary conditions, you will find that

$$V_{in} = \sum_l A_l r^l P_l(\cos\theta) \text{ and } V_{out} = -E_0 r\cos\theta \sum_l \frac{D_l}{r^{l+1}} P_l(\cos\theta)$$

Using the remaining boundary conditions, Fourier's Trick and the orthogonality of the Legendre Polynomials, we can find:

$$V_{in} = -\frac{3E_0}{\varepsilon_r+2} z \text{ and } V_{out} = -E_0 r\cos\theta + \frac{\varepsilon_r-1}{\varepsilon_r+2} R^3 E_0 \frac{\cos\theta}{r^2}$$

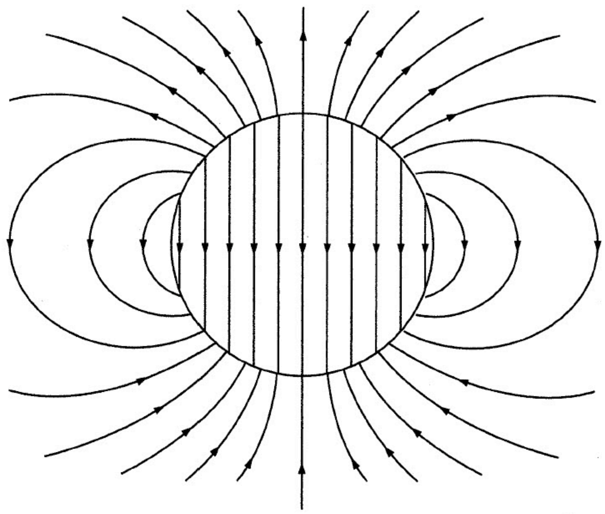
(c) Find the electric field $\vec{E}$ and polarization $\vec{P}$ inside the dielectric sphere.

$$\vec{E}_{in} = -\vec{\nabla} V_{in} = -\partial_z V_{in}\hat{z} = -\frac{3E_0}{\varepsilon_r+2}\hat{z}$$

and for a linear dielectric:

$$\vec{P} = \varepsilon_0\chi_e\vec{E} = \frac{\varepsilon_r-1}{\varepsilon_r+2}\varepsilon_0\chi_e E_0\hat{z}$$

(d) Sketch the electric field lines for all regions of this setup.



(e) Find the bound volume charge density ρ_b, and all the bound surface charge densities σ_b.

Since there are no free charge inside the sphere

$$\rho_b = 0.$$

The bound surface charge is

$$\sigma_b = \vec{P} \cdot \hat{n} = \frac{E_0(\varepsilon_r - 1)}{\varepsilon_r + 2} \varepsilon_0 \chi_e \cos\theta.$$

3. Inductance

(a) In the quasistatic approximation, find the induced electric field as a function of distance s from the wire. In the quasistatic approximation, the magnetic field of a wire is

$$B = \frac{\mu_0 I}{2\pi s}.$$

Using Faraday's law, we can find for the electric field:

$$\oint \vec{E}\, d\vec{l} = E(s_0)l - E(s)l = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a} = -\frac{\mu_0 l}{2\pi}\frac{dI}{dt}\int_{s_0}^{s}\frac{1}{s'}\,ds'$$
$$\Leftrightarrow \vec{E}(s) = -\left[\frac{\mu_0 I_0 \omega}{2\pi}\ln s \sin(\omega t) + K\right]\hat{z} = \left[6\times 10^{-6}\sin(\omega t) + K\right]\hat{z}.$$

(b) Is your answer valid for the limit $s \to \infty$? Explain your answer. No, since $\ln s$ diverges for large s The quasistatic approximation only holds for $s \ll ct$.

(c) Find the self-inductance L of the rectangular coil.

The magnetic field inside a toroid is given by

$$B = \frac{\mu_0 N I}{2\pi s}.$$

So, the flux through a single turn is

$$\Phi = \int \vec{B}\cdot d\vec{a} = \frac{\mu_0 N I}{2\pi} h \int_a^b \frac{1}{s}\,ds = \frac{\mu_0 N I h}{2\pi}\ln\frac{b}{a}.$$

Which means that the total flux is N times this, and the self-inductance is

$$\Phi = LI \Leftrightarrow L = \frac{\mu_0 N^2 h}{2\pi}\ln\frac{b}{a}.$$

(d) In the quasi-static approximation, what emf is induced in the coil?

In the quasistatic approximation:

$$\vec{B} = \frac{\mu_0}{2\pi s}\hat{\phi}.$$

So,

$$\phi_1 = \frac{\mu_0 I}{2\pi}\int_a^b \frac{1}{s} h\,ds = \frac{\mu_0 I h}{2\pi}\ln\frac{b}{a}.$$

This is the flux through only one turn, so the total flux is N times Φ_1:

$$\Phi = \frac{\mu_0 N h}{2\pi} \ln\frac{b}{a} I_0 \cos(\omega t).$$

So,

$$\mathscr{E} = -\frac{d\phi}{dt} = \frac{\mu_0 N h}{2\pi} \ln\frac{b}{a} I_0 \omega \sin(\omega t) = 2.61 \times 10^{-4} \sin(\omega t) \text{ V},$$

using

$$\omega = 377\frac{1}{s}.$$

(e) Find the current $I(t)$ in the resistor R.

$$I_r = \frac{\mathscr{E}}{R} = 5.22 \times 10^{-7} \sin\omega t \text{ A}.$$

(f) Calculate the back emf in the coil, due to the current $I(t)$. The back emf $\mathscr{E}_b$ is given by:

$$\mathscr{E}_b = -L\frac{dI_r}{dt};$$

Now use the self-inductance of the square coils calculated in part (a):

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln\frac{b}{a} = 1.39 \times 10^{-3} \text{ H}.$$

Therefore,

$$\mathscr{E}_b = -2.74 \times 10^{-7} \cos(\omega t) \text{ V}.$$

4. Momentum of Electromagnetic Fields

(a) What is the Poynting vector $\vec{S}$?

The Poynting vector is given by

$$\vec{S} = \frac{1}{\mu_0}\vec{E}_0 \times \vec{B}_0$$

The electric field of a parallel plate capacitor is given by

$$\vec{E} = -\frac{\sigma}{\varepsilon_0}\hat{z}$$

So

$$\vec{S} = \frac{\sigma B_0}{c^2\mu_0\varepsilon_0}\hat{x}$$

(b) What is the momentum density of the electromagnetic field? The momentum density is given by

$$\vec{p} = \frac{1}{c^2}\vec{E}_0 \times \vec{H}_0 \Rightarrow \vec{p} = \sigma B_0\hat{x}$$

(c) What is the total momentum of the electromagnetic field? The total momentum is given by

$$\vec{P}_{total} = \int \vec{p}\, dV = QB_0h\hat{x}$$

(d) What is the impulse $\Delta\vec{p}$ in time Δt experienced by each plate, as derived from the induced electric field? How does this compare to the field momentum derived in part c)?

Now, we consider a closed loop in the $x-z$-plane of width a and height h. The magnetic flux through this loop is:

$$\int \vec{B}_0\, d\vec{a} = B_b ah\hat{y}$$

and for the induced electric field:

$$\oint \vec{E}_{ind}\, d\vec{l} = -\partial_t \int \vec{B}_0\, d\vec{a} = -\frac{B_0 ah}{\Delta t}.$$

Taking the closed loop integral:

$$\oint \vec{E}_{ind}\, d\vec{l} = E_{ind} 2a = -\frac{B_0 ah}{\Delta t} \Rightarrow |E_{ind}| = -\frac{B_0 h}{2\Delta t}$$

The forces on the plates are now:

$$\vec{F} = q\vec{E} \text{ with } \vec{E}_{ind} = E_{ind}\hat{x} \text{ at the top plate and } \vec{E}_{ind} = -E_{ind}\hat{x} \text{ at the bottom plate.}$$

So, the force on both plates with Q on the top plate and $-Q$ on the bottom plate, is given by:

$$\vec{F} = Q\frac{B_0 h}{2\Delta t}\hat{x}$$

To find the momentum $\vec{p}$:

$$\vec{p} = \int \vec{F}\, dt = \frac{QB_0 h}{2}\hat{x}$$

So, each plate receives half of the momentum stored in the fields.

5. Dipole Radiation

(a) Find the electric field $\vec{E}(r,t)$ and magnetic field $\vec{B}(r,t)$ to leading order in powers of $\left(\frac{1}{r}\right)$ in terms of $\ddot{p}_0(t_r)$ [i.e. the second time derivative of $p_0(t)$ evaluated at the retarded time, t_r].

To find the electric field, use

$$\vec{E} = \vec{\nabla}V - \partial_t\vec{A}$$

Since we only consider terms of powers $\left(\frac{1}{r}\right)$ in terms of $\ddot{p}_0(t_r)$, we only deal with the last term of $V(\vec{r},t)$.

$$\begin{aligned}\vec{\nabla}V &= \tfrac{1}{4\pi\varepsilon_0}\partial_r\tfrac{\hat{r}\dot{\vec{p}}}{cr}\hat{r} = \tfrac{1}{4\pi\varepsilon_0}\tfrac{1}{cr}\left[\partial_r\left(\hat{r}\dot{\vec{p}}(t_r)\right)\right]\hat{r}\\ &= \tfrac{1}{4\pi\varepsilon_0}\tfrac{\hat{r}}{cr}\tfrac{\partial}{\partial t_r}\left(\hat{r}\dot{\vec{p}}(t_r)\right)\tfrac{\partial t_r}{\partial r}\\ &= -\tfrac{1}{4\pi\varepsilon_0}\tfrac{\hat{r}\ddot{\vec{p}}}{c^2 r}\hat{r}\end{aligned}$$

Now,

$$\partial_t\vec{A} = \tfrac{\mu_0}{4\pi}\tfrac{\partial}{\partial t_r}\left(\tfrac{\dot{\vec{p}}(t_r)}{r}\right) = \tfrac{\mu_0}{4\pi}\tfrac{\ddot{\vec{p}}(t_r)}{r}$$

Finally,

$$\vec{E} = \tfrac{1}{4\pi\varepsilon_0 c^2}\tfrac{\left(\hat{r}\cdot\ddot{\vec{p}}\right)\hat{r}-\ddot{\vec{p}}(t_r)}{r}$$

The magnetic field$\vec{B}$ can be found, as

$$\vec{B} = \vec{\nabla}\times\vec{A} = \tfrac{\mu_0}{4\pi}\vec{\nabla}\times\tfrac{\dot{\vec{p}}(t_r)}{r}$$

use the vector identity

$$\begin{aligned}\vec{\nabla}\times f\vec{v} &= f\left(\vec{\nabla}\times\vec{v}\right) = \left(\vec{v}\times\vec{\nabla}f\right)\\ \vec{\nabla}\times\tfrac{1}{r}\dot{\vec{p}} &= \tfrac{1}{r}\left(\vec{\nabla}\times\dot{\vec{p}}\right) - \dot{\vec{p}}\times\vec{\nabla}\left(\tfrac{1}{r}\right)\end{aligned}$$
and $\vec{\nabla}\left(\frac{1}{r}\right) = \frac{1}{r^2}\hat{r}$,which can be neglected.

Now,

$$\begin{aligned}&\vec{\nabla}\times\dot{\vec{p}} = \vec{\nabla}\times\hat{p}\dot{p} = -\hat{p}\times\vec{\nabla}\dot{p}\\ &\text{with } \vec{\nabla}\dot{p} = \ddot{p}\vec{\nabla}(t_r) = \ddot{p}\left(-\tfrac{\hat{r}}{c}\right)\end{aligned}$$

So, finally:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{1}{rc}\left(\hat{p}\times\ddot{p}\hat{r}\right) = -\frac{\mu_0}{4\pi rc}\left(\hat{r}\times\ddot{\vec{p}}\right)$$

(b) Assume that $\vec{p}(t_r) = p_0(t)\hat{z}$. Show that $\vec{E}(\vec{r},t) = E(\vec{r},t)\hat{\theta}$ and $\vec{B}(\vec{r},t) = B(\vec{r},t)\hat{\phi}$. Find expressions for $E(\vec{r},t)$ and $B(\vec{r},t)$.

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 c^2}\frac{\ddot{p}((\hat{r}\cdot\hat{z})\hat{r}-\hat{z})}{r} = \frac{\ddot{p}}{4\pi\varepsilon_0 c^2}\frac{\cos\theta\hat{r}-\hat{z}}{r}$$

using

$$\hat{z} = \cos\theta\hat{r} - \sin\theta\,\hat{\theta},$$

we finally find for the electric field:

$$\vec{E} = \frac{\ddot{p}}{4\pi\varepsilon_0 c^2}\frac{\sin\theta\hat{\theta}}{r}$$

Similarly for the magnetic field:

$$\vec{B} = -\frac{\mu_0}{4\pi cr}\ddot{p}\left(\hat{r}\times\hat{z}\right) = \frac{\mu_0}{4\pi cr}\ddot{p}\sin\theta\,\hat{\phi}.$$

(c) Find the power radiated to infinity by this time dependent charge distribution.

The total power is given by:

$$P = \int\vec{S}d\vec{a}$$

(d) Calculate the Poynting vector $\vec{S}$.

So, we need to find the Poynting vector:

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)\\ &= \frac{\mu_0}{16\pi^2 c}\frac{\ddot{p}_0^2}{r^2}\sin^2\theta\,\hat{r}\end{aligned}$$

So, the total power is then:

$$\begin{aligned}P = \int\vec{S}d\vec{a} &= \frac{\mu_0\ddot{p}_0^2}{16\pi^2 c}\int\frac{\sin^2\theta}{r^2}r^2\sin\theta\,d\theta d\phi\\ &= \frac{\mu_0\ddot{p}_0^2}{8\pi c}\int\sin^3\theta\,d\theta\\ &= \frac{\mu_0\ddot{p}_0^2}{6\pi c}\end{aligned}$$