University of Illinois at Chicago
Department of Physics

Electromagnetism
Qualifying Examination

January 6, 2011

9.00 am - 12.00 pm

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted towards the exam's total score.

Various equations, constants, etc. are provided on the last page of the exam.

1. Thomson Model of an Atom

One very simplistic way of describing an atom is to use a positive volume charge distribution $\rho(r)$ (created by the nucleus) which surrounds the electrons and balance their negative charges.

For the first part of the problem, only consider the positive volume charge distribution $\rho(r)$:

$$\rho(r) = \begin{cases} \rho_0 & : \quad r \leq R \\ 0 & : \quad r > R \end{cases}$$

(a) Find the corresponding electrostatic potential, V, inside and outside the atom. Do not pay any attention to the contributions of the electrons.

(b) Using your results from part (a), find the electrostatic vector-field, $\vec{E}$, inside and outside the atom.

Now consider a single electron inside this atom, moving under the influence of the potential $\rho(r)$.

(c) Explain why the electron will oscillate inside the atom, described by the charge distribution $\rho(r)$.

(d) Find the frequency of this oscillation.

2. Dielectric Sphere

A sphere of homogeneous dielectric material with permittivity ε and radius R is placed in an otherwise uniform electric field $\vec{E}_0$. The external electric field far away from the sphere is given by $\vec{E}_0 = E_0\hat{z}$.

(a) Determine the boundary conditions for the given setup.

(b) Find the potential inside and outside the dielectric sphere. Explain your approach!

(c) Find the electric field $\vec{E}$ and polarization $\vec{P}$ inside the dielectric sphere.

(d) Sketch the electric field lines for all regions of this setup.

(e) Find the bound volume charge density ρ_b, and all the bound surface charge densities σ_b.

3. Inductance

Consider an alternating current, $I(t)$, flowing down a straight wire:

$$I(t) = I_0 \cos \omega t$$

(a) In the quasistatic approximation, find the induced electric field as a function of distance, s, from the wire.

(b) Is your solution to part (a) valid for the limit $s \to \infty$? Explain your answer.

The wire runs parallel to the axis of a coil with rectangular cross-section which is connected to a resistor R. The wire is at a distance d from the coil of height and width h and N turns.

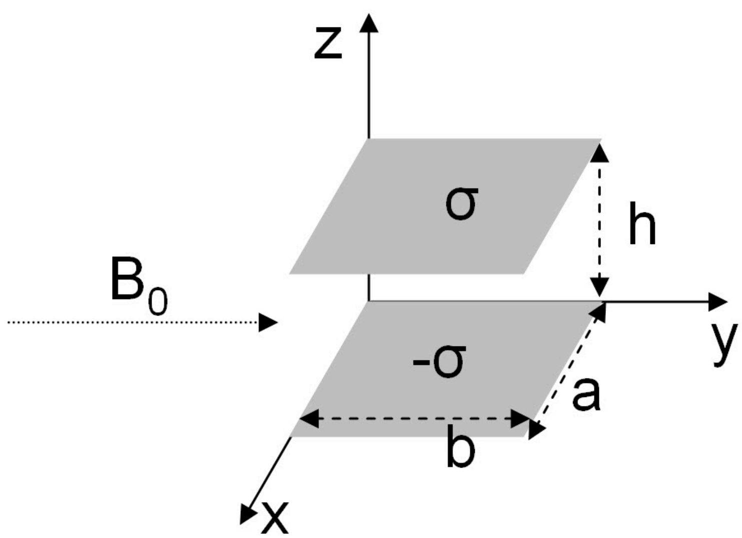
(c) Find the inductance L of the rectangular coil, in terms of N, h, a, and b.

(d) In the quasi-static approximation, what *emf* is induced in the coil?

(e) Calculate the back *emf* in the coil, due to the current $I(t)$.

4. Momentum of Electromagnetic Fields

Two non-conducting plates, both parallel to the $x-y$ plane, extend over the region $0 \leq x \leq a$ and $0 \leq y \leq b$. One plate is located at $z = 0$ and has a uniform charge density $-\sigma$. The second plate is located at $z = h$ and has a uniform charge density σ. Assume that the distance h between the plates is much smaller than their length a, and width b, so that edge effects can be ignored. There is a uniform magnetic field $\vec{B}_0 = B_0\hat{y}$.



(a) What is the Poynting vector $\vec{S}$?

(b) What is the momentum density of the electromagnetic field?

(c) What is the total momentum of the electromagnetic field?

Now the magnetic field $\vec{B}_0$ is turned off in a time Δt.

(d) What is the impulse $\Delta\vec{p}$ in time Δt experienced by each plate, as derived from the induced electric field? How does this compare to the field momentum derived in part (c)?

5. Dipole Radiation

Consider some time-dependent charge distribution of finite extent, $\rho(r,t)$, whose time dependent dipole moment is given by $\vec{p}(t) = p_0(t)\hat{p}$. In a region of space, the scalar and vector potentials established by this charge distribution are given by:

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r} + \frac{\hat{r}\cdot\vec{p}(t_r)}{r^2} + \frac{\hat{r}\cdot\dot{\vec{p}}(t_r)}{cr}\right]$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{\dot{\vec{p}}(t_r)}{r}$$

where $\dot{\vec{p}}$ is the time derivative of the dipole moment, and $t_r = t - \frac{r}{c}$ is the retarded time.

(a) Find the electric field $\vec{E}(r,t)$ and magnetic field $\vec{B}(r,t)$ to first order in $\left(\frac{1}{r}\right)$ in terms of $\ddot{p}_0(t_r)$ [i.e. the second time derivative of $p_0(t)$ evaluated at the retarded time, t_r].

(b) Assume that $\vec{p}(t_r) = p_0(t)\hat{z}$. Show that $\vec{E}(\vec{r},t) = E(\vec{r},t)\hat{\theta}$ and $\vec{B}(\vec{r},t) = B(\vec{r},t)\hat{\phi}$. Find expressions for $E(\vec{r},t)$ and $B(\vec{r},t)$.

(c) Calculate the Poynting vector $\vec{S}$.

(d) Derive explicitly the power radiated to infinity by this time dependent charge distribution.

Equations and Constants

$\hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$
$\hat{y} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}$
$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

$V(r) = \sum\limits_l \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$,
where $P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)$ and
$\int\limits_0^\pi P_l(\cos\theta)P_m(\cos\theta)\sin\theta\,d\theta = \begin{cases} 0 & if \quad m \neq l \\ \frac{2}{2l+1} & if \quad m = l \end{cases}$

$P_0(x) = 1$; $P_1(x) = x$; $P_2(x) = \frac{3x^2-1}{2}$; $P_3(x) = \frac{5x^3-3x}{2}$

$\vec{\nabla}\cdot\vec{v} = \partial_x v_x + \partial_y v_y + \partial_z v_z$ (cartesian);
$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\partial_r\left(r^2 v_r\right) + \frac{1}{r\,sin\theta}\partial_\theta\left(r\,sin\theta v_\theta\right) + \frac{1}{r\,sin\theta}\partial_\phi v_\phi$ (spherical)
$d\vec{a} = s\,d\phi\,dz\,\hat{r} + ds\,dz\,\hat{\phi} + s\,ds\,d\phi\,\hat{z}$ (cylindrical);
$d\vec{a} = r^2\sin\theta\,d\theta\,d\phi\,\hat{r} + r\,dr\,d\phi\,\hat{\theta} + r\sin\theta\,dr\,d\theta\,\hat{\phi}$ (spherical)

$\int\sin^3(ax)\,dx = -\frac{1}{a}\cos(ax) + \frac{1}{3a}\cos^3(ax)$