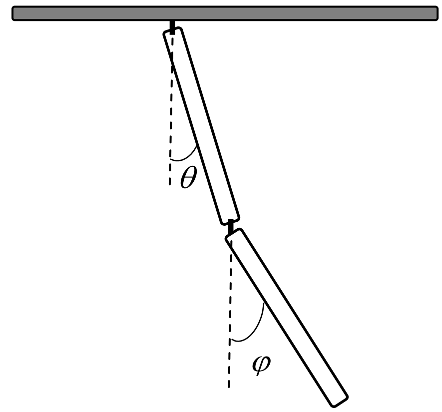
University of Illinois at Chicago
Department of Physics

Classical Mechanics
Qualifying Examination

January 4, 2011
9.00 am – 12:00 pm

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted toward the exam's total score.

Problem 1.



Two identical rods of mass *m* and length *l* are connected to the ceiling and together vertically by small flexible pieces of string. The system then forms a physical double pendulum. Find the frequencies of the normal modes of this system for small oscillations around the equilibrium position. Describe the motion of each of the normal modes.

Solution:

Let $\theta\,(\varphi)$ be the angle of the top (bottom) rod with vertical.

$$T = \frac{1}{2}\left(m(\frac{l}{2}\dot{\theta})^2 + \frac{1}{12}ml^2\dot{\theta}^2 + m(l\dot{\theta} + \frac{l}{2}\dot{\varphi})^2 + \frac{1}{12}ml^2\dot{\varphi}^2 \right)$$

$$U = mg\frac{l}{2}(1-\cos\theta) + mg\left(\frac{3}{2}l - \left(l\cos\theta + \frac{1}{2}\cos\varphi\right)\right) \approx mgl\left(\frac{\theta^2}{4} + \left(\frac{\theta^2}{2} + \frac{\varphi^2}{4}\right)\right)$$

$$L = T - U = \frac{4}{6}ml^2\dot{\theta}^2 + \frac{ml^2}{2}\dot{\theta}\dot{\varphi} + \frac{1}{6}ml^2\dot{\varphi}^2 - \frac{mgl}{4}(3\theta^2 + \varphi^2)$$

The Lagrange's equations are then given by

$$\frac{1}{2}\left(\frac{8}{3}l\ddot{\theta} + l\ddot{\varphi} + \frac{2}{3}g\theta\right) = 0 \qquad \frac{1}{2}\left(\frac{8}{3}l\ddot{\theta} + \frac{2}{3}l\ddot{\varphi} + \frac{1}{2}g\varphi\right) = 0$$

$$\frac{1}{2}\left(\frac{8}{3}\ddot{\theta} + \ddot{\varphi} + \frac{2}{3}\omega_0^2\theta\right) = 0 \qquad \frac{1}{2}\left(\ddot{\theta} + \frac{2}{3}\ddot{\varphi} + \frac{1}{2}\omega_0^2\varphi\right) = 0, \text{ where } \omega_0^2 = \frac{g}{l}$$

Assuming small oscillations with $\theta = A\cos\omega t$ and $\varphi = B\cos\omega t$ gives

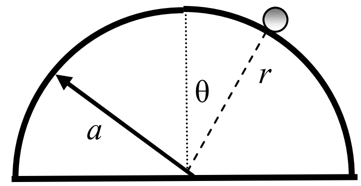
$$\begin{pmatrix} \frac{3}{2}\omega_0^2 - \frac{4}{3}\omega^2 & -\frac{\omega^2}{2} \\ -\frac{\omega^2}{2} & \frac{1}{2}\omega_0^2 - \frac{1}{3}\omega^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0$$

, which yields normal mode frequencies of

$$\omega^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\omega_0^2 = \begin{cases} 5.27\omega_0^2 \\ 0.73\omega_0^2 \end{cases}, \text{ and } \begin{cases} B = \left(\frac{-2\sqrt{7}}{3} - \frac{1}{3}\right)A = -2.10A \\ B = \left(\frac{2\sqrt{7}}{3} - \frac{1}{3}\right)A = -1.43A \end{cases}$$

Problem 2.

The particle is sliding down from the top of the hemisphere of radius a. Find: a) normal force exerted by the hemisphere on the particle; b) angle with respect to the vertical at which the particle will leave the hemisphere.



a) The equation of constraint is $f(r,\theta) = r - a = 0$

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \qquad V = mgr\cos\theta$$

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda\frac{\partial f}{\partial r} = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0 \qquad \frac{\partial f}{\partial r} = 1 \qquad \frac{\partial f}{\partial \theta} = 0$$

Thus $mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda = 0$

$$mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0$$

Now $r = a$, $\dot{r} = \ddot{r} = 0$ so

$$ma\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

$$mga\sin\theta - ma^2\ddot{\theta} = 0$$

$$\ddot{\theta} = \frac{g}{a}\sin\theta \qquad \text{and} \qquad \ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$

so $\int \dot{\theta}\, d\dot{\theta} = \frac{g}{a}\int \sin\theta\, d\theta$ or $\frac{\dot{\theta}^2}{2} = -\frac{g}{a}\cos\theta + \frac{g}{a}$

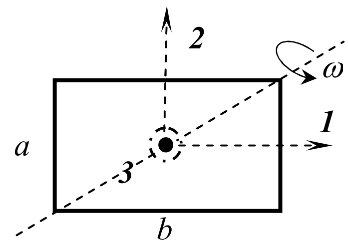
hence, $\lambda = mg\left(3\cos\theta - 2\right)$

b) and when $\lambda \to 0$ particle falls off hemisphere at

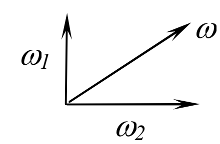
$$\theta_\circ = \cos^{-1}\left(\frac{2}{3}\right)$$

Problem 3.

A uniform rectangular plane lamina of mass m and dimensions a and b (assume $b > a$) rotates with the constant angular velocity ω about a diagonal. Ignoring gravity, find: a) principal axes and moments of inertia; b) angular momentum vector in the body coordinate system; c) external torque necessary to sustain such rotation.



a) $I_1 = \frac{ma^2}{12} \qquad I_2 = \frac{mb^2}{12} \qquad I_3 = I_1 + I_2 = \frac{m(a^2+b^2)}{12}$



b) $\omega_1 = \frac{\omega b}{(a^2+b^2)^{1/2}} \qquad \omega_2 = \frac{\omega a}{(a^2+b^2)^{1/2}} \qquad \omega_3 = 0$

$$\vec{L} = I_1\omega_1\hat{e}_1 + I_2\omega_2\hat{e}_2 + I_3\omega_3\hat{e}_3 = \left(\frac{ma^2}{12}\right)\frac{\omega b}{(a^2+b^2)^{1/2}}\hat{e}_1 + \left(\frac{mb^2}{12}\right)\frac{\omega a}{(a^2+b^2)^{1/2}}\hat{e}_2 + 0\hat{e}_3$$

$$\vec{L} = \frac{mab\omega}{12(a^2+b^2)^{1/2}}(a,b,0)$$

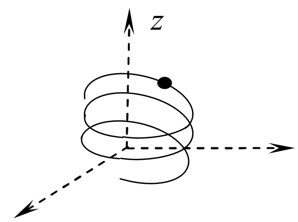
c) In body coordinate system $\vec{\omega} = const$

$$\vec{\tau} = \frac{d\vec{L}}{dt} + \vec{\omega}\times\vec{L} = \vec{\omega}\times\vec{L}$$

$$\vec{\tau} = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & 0 \\ L_1 & L_2 & 0 \end{vmatrix} = (\omega_1 L_2 - \omega_2 L_1)\hat{e}_3$$

$$\vec{\tau} = \frac{mab\omega^2}{12(a^2+b^2)^{1/2}}(b^2-a^2)\hat{e}_3$$

Problem 4.



A particle of mass m moves frictionless under the influence of gravity along the helix $z = k\theta$, $r = const$, where k is a constant, and z is vertical. Find: a) the Lagrangian; b) the Hamiltonian. Determine: c) equations of motion.

In cylindrical coordinates the kinetic energy and the potential energy of the spiraling particle are expressed by

$$\left.\begin{aligned} T &= \frac{1}{2} m \left[\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 \right] \\ U &= mgz \end{aligned}\right] \quad (1)$$

Therefore, if we use the relations,

$$\left.\begin{aligned} &z = k\theta \qquad \text{i.e., } \dot{z} = k\dot{\theta} \\ &r = \text{const.} \end{aligned}\right] \quad (2)$$

the Lagrangian becomes

$$L = \frac{1}{2} m \left[\frac{r^2}{k^2} \dot{z}^2 + \dot{z}^2 \right] - mgz \quad (3)$$

Then the canonical momentum is

$$p_z = \frac{\partial L}{\partial \dot{z}} = m \left[\frac{r^2}{k^2} + 1 \right] \dot{z} \quad (4)$$

or,

$$\dot{z} = \frac{p_z}{m \left[\frac{r^2}{k^2} + 1 \right]} \quad (5)$$

The Hamiltonian is

$$H = p_z \dot{z} - L = p_z \frac{p_z}{m \left[\frac{r^2}{k^2} + 1 \right]} - \frac{p_z^2}{2m \left[\frac{r^2}{k^2} + 1 \right]} + mgz \quad (6)$$

or,

$$H = \frac{1}{2} \frac{p_z^2}{m \left[\frac{r^2}{k^2} + 1 \right]} + mgz \quad (7)$$

Now, Hamilton's equations of motion are

$$-\frac{\partial H}{\partial z} = \dot{p}_z; \quad \frac{\partial H}{\partial p_z} = \dot{z} \tag{8}$$

so that

$$-\frac{\partial H}{\partial z} = -m\,g = \dot{p}_z \tag{9}$$

$$\frac{\partial H}{\partial p_z} = \frac{p_z}{m\left[\frac{r^2}{k^2}+1\right]} = \dot{z} \tag{10}$$

Taking the time derivative of (10) and substituting (9) into that equation, we find the equation of motion of the particle:

$$\boxed{\ddot{z} = \frac{g}{\left[\frac{r^2}{k^2}+1\right]}} \tag{11}$$

Problem **5.**

A particle of mass m is bound by the linear potential $U = kr$, where $k = const$. Find:

a) For what energy and angular momentum will the orbit be a circle of radius r about the origin?
b) What is the frequency of this circular motion?
c) If the particle is slightly disturbed from this circular motion, what will be the frequency of small oscillations?

The force acting on the particle is $\overline{F} = -\frac{dU}{dr}\hat{r} = -k\hat{r}$

a) For particle moving on a circular orbit of radius r: $m\omega^2 r = k$, i.e. $\omega^2 = \frac{k}{mr}$

The energy of the particle is then $E = kr + \frac{mv^2}{2} = kr + \frac{m\omega^2 r^2}{2} = \frac{3kr}{2}$

Its angular momentum about the orbit is $L = m\omega r^2 = mr^2\sqrt{\frac{k}{mr}} = \sqrt{mkr^3}$

b) The angular frequency of circular motion is $\omega = \sqrt{\frac{k}{mr}}$.

c) The effective potential is $U_{eff} = kr + \frac{L^2}{2mr^2}$.

The radius r_0 of the stationary circular motion is given by

$$\left(\frac{dU_{eff}}{dr}\right)_{r=r_0} = k - \frac{L^2}{2mr_0^3} = 0, \text{ i.e. } r_0 = \left(\frac{L^2}{mk}\right)^{1/3}$$

As $\left(\frac{d^2U_{eff}}{dr^2}\right)_{r=r_0} = \left.\frac{3L^2}{2mr^4}\right|_{r=r_0} = \frac{3L^2}{m}\left(\frac{mk}{L^2}\right)^{4/3} = 3k\left(\frac{mk}{L^2}\right)^{1/3}$, the angular frequency of oscillations about r_0, if it is slightly disturbed from the stationary circular motion, is

$$\omega_r = \sqrt{\frac{1}{m}\left(\frac{d^2U_{eff}}{dr^2}\right)_{r=r_0}} = \sqrt{3k\left(\frac{mk}{L^2}\right)^{1/3}} = \sqrt{\frac{3k}{mr_0}} = \sqrt{3}\omega_0,$$

where ω_0 is the angular frequency of the stationary circular motion.