(1) $P_2 = \frac{1}{2}(3\cos^2\theta - 1)$ ; $P_0 = 1$ ;

(a) $$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{2P_2+1}{3}\right) - 1 = \frac{4}{3}P_2 - \frac{1}{3}P_0$$

$$\Phi(a,\theta) = \frac{V_0}{3}\left(4P_2(\cos\theta) - P_0(\cos\theta)\right)$$

$\Downarrow$ by inspection

$$\boxed{\Phi(r,\theta) = \frac{V_0}{3}\left(\frac{4a^3}{r^3}P_2(\cos\theta) - \frac{a}{r}\right)}$$

(b) $$\Phi = \frac{Q}{4\pi\varepsilon_0 r} + O\left(\frac{1}{r^2}\right)$$

$$\Rightarrow \frac{Q}{4\pi\varepsilon_0} = -\frac{V_0 a}{3} \Rightarrow \boxed{Q = -\frac{4\pi\varepsilon_0}{3}V_0 a}$$

② (a) $U = \int d^3r\, \Phi \rho = \int d^3r \frac{e_0}{r}\left(-\frac{e_0}{\pi a^3} e^{-2r/a}\right)$

$$= -\frac{4\pi e_0^2}{\pi a^3}\int_0^\infty dr \cdot r e^{-2r/a} = -\frac{4\pi e_0^2}{\pi a^3}\cdot\frac{a^2}{4} = \boxed{-\frac{e_0^2}{a}}.$$

(b) $U = \Phi(\vec{0})\cdot e_0 \Rightarrow \Phi(\vec{0}) = \frac{U}{e_0} = \boxed{-\frac{e_0}{a}}$.

②

(c) Use Gauss' law:

$$\varepsilon_0 4\pi r^2 E_r = \underset{\substack{\uparrow \\ \text{proton}}}{e_0} + 4\pi\int_0^r dr' r'^2\left(-\frac{e_0}{\pi a^3}\right)e^{-2r'/a}$$

$$= e_0 - 4\pi\frac{e_0}{\pi a^3}\cdot\left(\frac{a}{2}\right)^3\cdot\int_0^{2r/a} dx\, x^2 e^{-x}$$

$$= e_0 - \frac{e_0}{2}\left[e^{-x}(-2-2x-x^2)\right]_0^{2r/a}$$

$$= e_0 - \frac{e_0}{2}\left[e^{-2r/a}\left(-2-\frac{4r}{a}-\frac{4r^2}{a^2}\right)-(-2)\right]$$

$$= e_0 e^{-2r/a}\left(1+\frac{2r}{a}+\frac{2r^2}{a^2}\right)$$

$$\boxed{\begin{aligned}&E_r = \frac{1}{4\pi\varepsilon_0}\frac{e_0}{r^2}e^{-2r/a}\left(1+\frac{2r}{a}+\frac{2r^2}{a^2}\right) > 0\\ &\vec{E} = E_r\hat{r} \quad - \boxed{\text{outward.}}\end{aligned}}$$

③ $\Phi_{in} = A r \cos\theta$ ; $\Phi_{out} = \left(\frac{B a^3}{r^2} - E_0 r\right)\cos\theta$

(a) Boundary conditions:

(i) $(E_t)_{in} = (E_t)_{out} \Rightarrow \phi_{in} = \phi_{out}$

$$\Downarrow$$

$$A a = B a - E_0 a \quad (*)$$

(ii) $(D_n)_{in} = (D_n)_{out} \Rightarrow \varepsilon \frac{\partial \phi_{in}}{\partial n} = \varepsilon_0 \frac{\partial \phi_{out}}{\partial n}$

$$\varepsilon A = -2\varepsilon_0 B - \varepsilon_0 E_0 \quad (**)$$

$$(*) \Rightarrow A = B - E_0$$

$$(**) \Rightarrow \varepsilon(B - E_0) = -2\varepsilon_0 B - \varepsilon_0 E_0$$

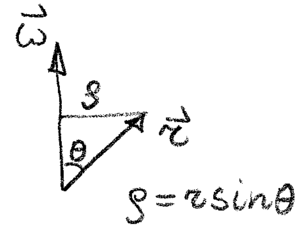
$$B = \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 \Rightarrow A = B - E_0 = -\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0$$

$$\boxed{\Phi_{in} = -\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 r \cos\theta \ ; \ \Phi_{out} = \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\frac{a^3}{r^2} - r\right) E_0 \cos\theta}$$

(b) $$\sigma_{out} = \varepsilon_0 \frac{\partial \Phi_{out}}{\partial n} = \varepsilon_0 \left.\frac{\partial \phi_{out}}{r\,\partial\theta}\right|_{\theta = \pi/2} = \boxed{\varepsilon_0 E_0 \left(1 - \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\left(\frac{a}{r}\right)^3\right)}$$

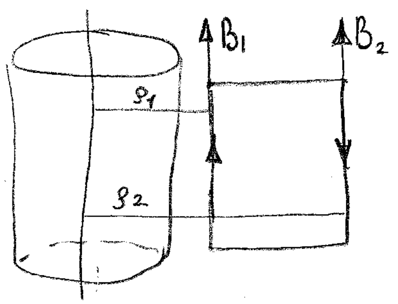
④

(a) $\vec{J} = \alpha \vec{v} = \alpha \vec{\omega} \times \vec{r} \quad \Rightarrow \quad J_\rho = J_\theta = 0$




$\boxed{J_\varphi = \alpha \omega \rho}$

(b) Ampere's law for the loop shown:




$(B_1 - B_2) \cdot \ell = 0$

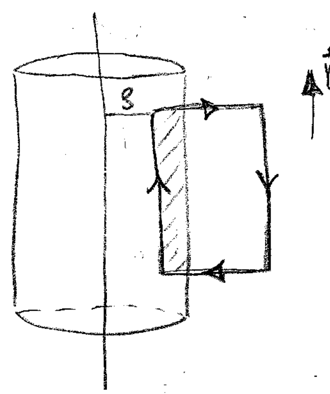
$\Downarrow$

$\vec{B}_1 = \vec{B}_2$

since $\vec{B}(\rho = \infty) = 0$

$\Rightarrow \boxed{\vec{B}(\rho) = 0 \text{ for } \rho > a}$

(c) Ampere's law for the loop shown:




$\vec{B} = \text{const} \Rightarrow \vec{B} = 0$

$B_z \cdot \ell = \mu_0 \ell \cdot \int_\rho^a d\rho' \cdot J_\varphi = \mu_0 \ell \alpha \omega \left( \frac{a^2}{2} - \frac{\rho^2}{2} \right)$

$\boxed{B_z = \mu_0 \frac{\alpha\omega}{2} (a^2 - \rho^2) \; ; \quad B_\rho = B_\varphi = 0}$

5

(a) $$\dot{\vec{p}} = \int d^3r\, \dot{\rho}\,\vec{r} = -\int d^3r\,(\vec{\nabla}\cdot\vec{J})\,\vec{r} = \int d^3r\, \vec{J}$$

$$= \oint d\vec{l}\, I = \int_0^{2\pi} \underbrace{a\,d\varphi\,\hat{\varphi}}_{d\vec{l}}\, I_0 \sin\varphi \cos\omega t$$

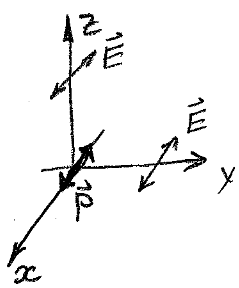
$$= I_0 a \cos\omega t \int_0^{2\pi} d\varphi \,(\underbrace{-\hat{x}\sin\varphi + \hat{y}\cos\varphi}_{\hat{\varphi}})\sin\varphi$$

$$= \boxed{-\pi I_0 a \cos\omega t \cdot \hat{x}}$$

(b) $$\vec{m} = \frac{1}{2}\int d^3r\, \vec{r}\times\vec{J} = \frac{1}{2}\oint I\,\vec{r}\times d\vec{l} = \frac{1}{2}\int_0^{2\pi} I\, \underbrace{\vec{r}\times\hat{\varphi}}_{=0}\, a\,d\varphi = 0$$

$$\Rightarrow \boxed{\dot{\vec{m}} = 0.}$$

(c)




| direction | polarization | axis of polarization |
|---|---|---|
| x | none | none |
| y | linear | x |
| z | linear | x |