Quantum Solutions

1. (a) First we show that $V(r) < V_c(r)$ for all $r > 0$. This amounts to showing that $1/r < \alpha/\ell n(1+\alpha r)$, or equivalently

$$\ell n(1+\alpha r) < \alpha r$$

which clearly holds. Next, we will use the *variational theorem*: The expectation value of the Hamiltonian operator in *any* state $|\psi\rangle$ is greater than or equal to the ground state energy of the system (with equality holding if and only if $|\psi\rangle$ is the exact ground state). Specifically, we take the expectation value of the Hamiltonian for our system

$$H = \frac{-\hbar^2}{2M}\nabla^2 + V(r)$$

in the Coulombic ground state $|\psi_c\rangle$. Since $V(r) < V_c(r)$ for all $r > 0$, we see that $\langle H\rangle_{|\psi_c\rangle}$ is less than the Coulomb ground state energy. But by the variational theorem $\langle H\rangle_{|\psi_c\rangle}$ is greater than the ground state energy for $V(r)$.

(b) We have that (after some algebra)

$$V(r) = -\frac{A}{r} - \alpha\,\frac{A}{2} - \alpha^2\frac{A\,r}{12} + \mathcal{O}(\alpha^3) \equiv V_c(r) + \alpha\,V_1(r) + \alpha^2 V_2(r) + \mathcal{O}(\alpha^3) \quad .$$

Thus, in order to obtain the ground state energy correct to order α^2, we must do perturbation theory about $V_c(r)$ to second-order in $V_1(r)$ and to first-order in $V_2(r)$. Since $V_1(r)$ is a constant, the associated first-order correction is just $\alpha\,V_1(r)$, and the associated second-order correction vanishes. The first-order correction from $V_2(r)$ gives $-\alpha^2 A\,\langle r\rangle/12$, where the expectation value is taken in the Coulomb ground state. Putting this all together we obtain

$$E_{\rm ground} = -\frac{MA^2}{2\hbar^2} - \alpha\,\frac{A}{2} - \alpha^2\frac{\hbar^2}{8M} + \mathcal{O}(\alpha^3) \quad ,$$

where the first term is the Coulombic ground state energy, and we have used the fact that $\langle r\rangle = 3\hbar^2/2MA$.

2. (a) The "equation of motion" for $\langle\,\Pi\,\rangle(t)$ is

$$\frac{d\,\langle\,\Pi\,\rangle(t)}{dt} = \frac{i}{\hbar}\langle\,[\,H,\Pi\,]\,\rangle = \frac{i}{\hbar}\langle\,[\,V(x),\Pi\,]\,\rangle \quad .$$

Here $H = p^2/2M + V(x)$ is the Hamiltonian operator of the system, and in the last equality we have used that $[\,p^2,\Pi\,] = 0$. Since by assumption $\langle\,\Pi\,\rangle(t)$ changes with time, we know that $[\,V(x),\Pi\,] \neq 0$. But this commutator is zero if $V(x)$ is symmetric.

(b) The eigenvalues of Π are ± 1, with symmetric $(+1)$ and anti-symmetric (-1) wavefunctions associated with the corresponding eigenvectors. Thus, $\langle\,\Pi\,\rangle(t) = 1$ means that the state $|\psi(t)\rangle$ has a symmetric wavefunction. A measurement of the energy will project this state onto the different energy eigenstates $|\psi_n\rangle$ $(n = 0, 1, 2, \ldots)$ with associated probabilities $|\langle\psi(t)|\psi_n\rangle|^2$. (Here $n = 0$ corresponds to the ground state, $n = 1$ to the first excited state, etc.) But these probabilities will be zero for odd n since $V(x)$ symmetric implies that wavefunction of $|\psi_n\rangle$ is anti-symmetric in these cases. In particular, $|\langle\psi(t)|\psi_1\rangle|^2 = 0$.

3. First note that the eigenvalues of H are $\pm\lambda$ independent of $\theta(t)$.

(a) Here we use the *sudden approximation*: The state of the system immediately after the change ($t = 0^+$) will be the same as the state of the system immediately before the change ($t = 0^-$). We have that

$$H(t = 0^-) = \lambda\big[\, |1\rangle\langle 1| - |2\rangle\langle 2| \,\big] \quad ,$$

so that the ground state at $t = 0^-$ is $|\psi_0\rangle = |2\rangle$ (remember that $\lambda > 0$). Thus, $|\psi(t = 0^-)\rangle = |\psi(t = 0^+)\rangle = |2\rangle$. But

$$H(t = 0^+) = \lambda\big[\, |1\rangle\langle 2| + |2\rangle\langle 1| \,\big] \quad ,$$

so that the ground state at $t = 0^+$ is given by $|\phi_0\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$. The probability that the system will be found in the ground state upon an energy measurement at time $t = 0^+$ is thus $|\langle\phi_0|\psi_0\rangle|^2 = 1/2$.

(b) Here we use the *adiabatic approximation*: If at time t_1 the system is in a non-degenerate eigenstate of a "slowly-varying" Hamiltonian $H(t)$, and if this energy level does not cross any other level as a function of t, then the state of the system at any later time t_2 is the eigenstate of $H(t_2)$ which is related by continuity to the relevant eigenstate of $H(t_1)$. Note that since the eigenvalues of $H(t)$ are $\pm\lambda$ for all t, we have that the energy levels are nondegenerate and do not cross. Moreover, the state of the system at time t_1 is the ground state of $H(t_1)$. Thus, in the adiabatic approximation the state of the system at time t_2 will be the ground state of $H(t_2)$, which is given by $\frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$.

4. (a) Since $\mathcal{H}$ is finite-dimensional, we are assured that $\mathrm{Tr}(C)$ exists and is given by $\mathrm{Tr}(C) = i\,\mathrm{Tr}([\,A,\,B\,]) = i\,\big[\mathrm{Tr}(AB) - \mathrm{Tr}(BA)\big] = 0$. But $\mathrm{Tr}(C)$ is simply the sum of the eigenvalues of C. Moreover, C is Hermitian (note the factor of i) and hence has only real eigenvalues. Since we know that $[\,A,\,B\,] \neq 0$, all of the eigenvalues of C cannot be zero. Hence the sum of the eigenvalues being zero and real implies that there is at least one positive and at least one negative eigenvalue. If the dimension of $\mathcal{H}$ is infinite, then $\mathrm{Tr}(C)$ may not exist and thus the above proof does not go through. An explicit counterexample to the claim is provided by letting $\mathcal{H}$ be the Hilbert space associated with a single particle in one spatial dimension, and setting $A = \hat{p}$ and $B = \hat{x}$ so that $C = \hbar I$ (where I is the identity operator on $\mathcal{H}$). In this case we see that all of the eigenvalues of C are positive.

(b) (i) Assume that A (say) has a zero eigenvalue with corresponding eigenvector $|\psi\rangle$. We then have $AB\,|\psi\rangle = BA\,|\psi\rangle = 0$. Hence $|\psi\rangle$ is an eigenvector of AB with zero eigenvalue. (ii) Conversely, assume that AB has a zero eigenvalue with corresponding eigenvector $|\phi\rangle$, that is, $AB\,|\phi\rangle = 0$. There are two possibilities here. First, we can have $B\,|\phi\rangle = 0$, in which case B has a zero eigenvalue. Second, it may be that $B\,|\phi\rangle \neq 0$. But then $B\,|\phi\rangle$ is an eigenvector of A with eigenvalue zero. Note that this proof did not use the finite-dimensionality of $\mathcal{H}$ in any way, so that it is valid in the infinite-dimensional case.

5. (a) Both the electron and the proton have spin 1/2. That is, $s_1 = s_2 = 1/2$. These may combine to either give either $s = 0$ or $s = 1$ (where $\vec{S} \equiv \vec{S}_1 + \vec{S}_2$). In the former case, the only way to obtain total angular momentum j is to have $\ell = j$. This gives $(2j + 1)$ linearly independent states. When $s = 1$ we can obtain total angular momentum j with $\ell = j - 1, j, j + 1$ if $j \geq 1$, while for $j = 0$ we can only use $\ell = j + 1 = 1$. Thus, the $s = 1$ case gives $3(2j + 1)$ linearly independent states for $j \geq 1$ and only $(2j + 1) = 1$ state for $j = 0$. Putting this all together, the total number of linearly independent states is $4(2j + 1)$ for $j \geq 1$, and $2(2j + 1) = 2$ for $j = 0$.

(b) The two linearly independent states with $j = 0$ are given by

$$\frac{1}{\sqrt{2}}\big[\,|1/2,1/2\rangle \otimes |1/2,-1/2\rangle - |1/2,-1/2\rangle \otimes |1/2,1/2\rangle\,\big] \otimes |0,0\rangle$$

and

$$\begin{aligned}\frac{1}{\sqrt{3}}\big\{\,&|1/2,1/2\rangle \otimes |1/2,1/2\rangle \otimes |1,-1\rangle + |1/2,-1/2\rangle \otimes |1/2,-1/2\rangle \otimes |1,1\rangle \\ &-\frac{1}{\sqrt{2}}\big[\,|1/2,1/2\rangle \otimes |1/2,-1/2\rangle + |1/2,-1/2\rangle \otimes |1/2,1/2\rangle\,\big] \otimes |1,0\rangle\,\big\} \quad .\end{aligned}$$

These are straightforwardly obtained using the standard techniques for calculating Clebsch-Gordon coefficients.