Quantum Problems

1. Consider a particle of mass M moving in the spherically symmetric potential $V(r) = -A\,\alpha/\ell n(1+\alpha\, r)$ in three spatial dimensions ($A, \alpha > 0$).

(a) Prove that the ground state energy of the system is strictly less than the corresponding ground state energy for the potential $V_c(r) = -A/r$ (for any $A, \alpha > 0$).

(b) Assuming α to be "small", expand $V(r)$ in a power series in α and calculate the ground state energy complete to order α^2.

2. Consider a particle moving in the potential $V(x)$ in one spatial dimension, and let Π be the parity operator.

(a) Assume that the time-evolving state of the system is such that $\langle\,\Pi\,\rangle(t_1) \neq \langle\,\Pi\,\rangle(t_2)$ for some times t_1 and t_2. Show that $V(x)$ is *not* a symmetric potential.

(b) Assume instead that $V(x)$ *is* symmetric and that $\langle\,\Pi\,\rangle(t) = 1$. If a measurement of the energy is performed at time t, show that the probability of finding the system in the first excited state is equal to zero.

3. Consider a quantum system with a two-dimensional Hilbert space $\mathcal{H}$. Let $\{|1\rangle, |2\rangle\}$ be an orthonormal basis for $\mathcal{H}$, and let the time-dependent Hamiltonian operator for the system be given by ($\lambda > 0$)

$$H(t) = \lambda\Big[\cos(\theta(t))\,|1\rangle\langle 1| + \sin(\theta(t))\,|1\rangle\langle 2| + \sin(\theta(t))\,|2\rangle\langle 1| - \cos(\theta(t))\,|2\rangle\langle 2|\,\Big] \quad .$$

(a) Assume that $\theta(t<0) = 0$ and $\theta(t>0) = \pi/2$. If the system is in the ground state at time $t = 0^-$, what is the probability that it will be found in the ground state upon an energy measurement at time $t = 0^+$?

(b) Assume instead that $\theta(t)$ is "slowly-varying", and that $\theta(t_1) = 0$ and $\theta(t_2) = \pi/2$. If the system is in the state $|2\rangle$ at time t_1, what is the state at time t_2?

4. Let A and B be two Hermitian operators on a finite-dimensional Hilbert space $\mathcal{H}$.

(a) Assume that A and B do not commute. Prove that the operator $C \equiv i\,[A, B]$ has (at least) one eigenvalue which is real and negative, and (at least) one eigenvalue which is real and positive. Is this result still true if $\mathcal{H}$ is infinite-dimensional?

(b) Assume instead that A and B commute. Prove that the operator AB has a zero eigenvalue if and only if either A or B has a zero eigenvalue. Is this result still true if $\mathcal{H}$ is infinite-dimensional?

5. The total angular momentum operator $\vec{J} = \vec{S}_1 + \vec{S}_2 + \vec{L}$ for a hydrogen atom has contributions from the spin of the electron ($\vec{S}_1$), the spin of the proton ($\vec{S}_2$), and the relative orbital angular between the electron and proton ($\vec{L}$).

(a) How many linearly independent states of the system are there with total angular momentum quantum number j? (Ignore radial excitations.)

(b) Write down a maximal set of linearly independent states with $j = 0$, each expressed as a linear combination of the states $|s_1, m_{s_1}\rangle \otimes |s_2, m_{s_2}\rangle \otimes |\ell, m_\ell\rangle$.