Problem 1

a) Let x_1, x_2, x_3 be the displacements of the masses along the x-axis.

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \frac{1}{\beta}\dot{x}_2^2 + \dot{x}_3^2\right)$$

$$U = \frac{1}{2}\alpha k x_1^2 + \frac{1}{2}\alpha k x_3^2 + \frac{1}{2}k\left(x_1 - x_2\right)^2 + \frac{1}{2}k\left(x_3 - x_2\right)^2$$

Then $\mathcal{L} = T - U$

b)

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial \mathcal{L}}{\partial x_i}$$

$$i = 1 \qquad m\ddot{x}_1 = -\alpha k x_1 - k\left(x_1 - x_2\right)$$

$$i = 2 \qquad \frac{m}{\beta}\ddot{x}_2 = k\left(x_1 - x_2\right) + k\left(x_3 - x_2\right) = k\left(x_1 - 2x_2 + x_3\right)$$

$$i = 3 \qquad m\ddot{x}_3 = -\alpha k x_3 - k\left(x_3 - x_2\right)$$

and letting $x_i = a_i e^{i\omega t}$ *one obtains*

$$i = 1 \qquad -\omega^2 a_1 = -\frac{\alpha k}{m}a_1 - \frac{k}{m}\left(a_1 - a_2\right) \qquad \Rightarrow \qquad \left(1+\alpha\right)a_1 - a_2 = \lambda a_1$$

$$i = 2 \qquad -\omega^2 a_2 = \frac{k\beta}{m}\left(a_1 - 2a_2 + a_3\right) \qquad \Rightarrow \qquad -\beta a_1 + 2\beta a_2 - \beta a_3 = \lambda a_2$$

$$i = 3 \qquad -\omega^2 a_3 = -\frac{\alpha k}{m}a_3 - \frac{k}{m}\left(a_3 - a_2\right) \qquad \Rightarrow \qquad -a_2 + \left(1+\alpha\right)a_3 = \lambda a_3$$

c) *The last result in (b) is reproduced by the matrix equation*

$$\begin{bmatrix} (1+\alpha) & -1 & 0 \\ -\beta & 2\beta & -\beta \\ 0 & -1 & (1+\alpha) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{So, we solve the sec. eq.} \quad \begin{vmatrix} (1+\alpha)-\lambda & -1 & 0 \\ -\beta & 2\beta-\lambda & -\beta \\ 0 & -1 & (1+\alpha)-\lambda \end{vmatrix} = 0$$

$$\left\{(1+\alpha)-\lambda\right\}\left\{\left[(1+\alpha)-\lambda\right]\left[2\beta-\lambda\right]-\beta\right\} + 1\left\{\left[(1+\alpha)-\lambda\right]\left[-\beta\right]\right\} = 0$$

$$\left\{(1+\alpha)-\lambda\right\}\left\{\left[(1+\alpha)-\lambda\right]\left[2\beta-\lambda\right]-2\beta\right\} = 0$$

and expanding $\left\{(1+\alpha)-\lambda\right\}\left\{\lambda^2 - \left(1+\alpha+2\beta\right)\lambda + 2\alpha\beta\right\} = 0$

Hence: $\lambda_o = 1+\alpha \qquad \lambda_\pm = \frac{1}{2}\left(1+\alpha+2\beta\right) \pm \frac{1}{2}\left[\left(1+\alpha+2\beta\right)^2 - 8\alpha\beta\right]^{1/2}$

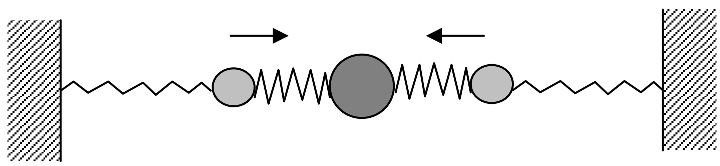
d) Method 1: We were told in the part (c) hint that λ_o should depend only on α. Hence, we can confirm that the $1 + \alpha$ eigen frequency, indeed, corresponds to the λ_o solution. From (b):

$$(1+\alpha)\boldsymbol{a}_1 - \boldsymbol{a}_2 = \lambda_o \boldsymbol{a}_1 \quad \Rightarrow \quad \boldsymbol{a}_2^{(o)} = 0$$

$$\textit{then} \quad -\beta\boldsymbol{a}_1 + 2\beta\boldsymbol{a}_2 - \beta\boldsymbol{a}_3 = \lambda_o \boldsymbol{a}_2 \quad \Rightarrow \quad -\beta\boldsymbol{a}_1^{(o)} - \beta\boldsymbol{a}_3^{(o)} = 0$$

$$\textit{Hence} \quad \boldsymbol{a}_1^{(o)} = -\boldsymbol{a}_3^{(o)}$$

Method 2: By symmetry $\boldsymbol{a}_1^{(o)} = -\boldsymbol{a}_3^{(o)}$. The ball and spring system is fully symmetric under reflection about the origin. The only way to satisfy this symmetry when $\boldsymbol{a}_2^{(o)} = 0$ *is to have* $\boldsymbol{a}_1^{(o)} = -\boldsymbol{a}_3^{(o)}$. This mode has the form,



e) From part (b):

$$(1+\alpha)\boldsymbol{a}_1 - \boldsymbol{a}_2 = \lambda_\pm \boldsymbol{a}_1 \quad \Rightarrow \quad (1+\alpha-\lambda_\pm)\boldsymbol{a}_1^\pm = \boldsymbol{a}_2^\pm$$

$$-\boldsymbol{a}_2 + (1+\alpha)\boldsymbol{a}_3 = \lambda_\pm \boldsymbol{a}_3 \quad \Rightarrow \quad (1+\alpha-\lambda_\pm)\boldsymbol{a}_3^\pm = \boldsymbol{a}_2^\pm$$

$$\textit{Hence} \quad \boldsymbol{a}_1^\pm = \boldsymbol{a}_3^\pm$$

The out of phase mode has the higher frequency. For the in-phase mode all atoms move in the same direction, so it has the heaviest effective mass and must have the lowest frequency.

Problem 2

$$H = \sum_i \dot{q}_i p_i - \mathcal{L}$$

Here $i = 3$, $q_1 = x$ $\quad q_2 = y$ $\quad q_3 = z$

$p_1 = p_x$ $\quad p_2 = p_y$ $\quad p_3 = p_z$

$$\mathcal{L} = K - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

$$H = \dot{x} p_x + \dot{y} p_y + \dot{z} p_z - \left[\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz\right]$$

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \qquad p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y} \qquad p_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{z}$$

a) $H = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz = E = K + V$

V is independent of $\dot{q}$, $K \propto \dot{q}^2$ & independent of time $\Rightarrow$ H = Total Mechanical Energy

b) $$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + mgz$$

c) $$\frac{\partial H}{\partial p_x} = \dot{x} \qquad \frac{\partial H}{\partial p_y} = \dot{y} \qquad \frac{\partial H}{\partial p_z} = \dot{z}$$

$$\frac{\partial H}{\partial x} = \dot{p}_x \qquad \frac{\partial H}{\partial y} = \dot{p}_y \qquad \frac{\partial H}{\partial z} = \dot{p}_z$$

$$\frac{p_x}{m} = \dot{x} \qquad \frac{p_y}{m} = \dot{y} \qquad \frac{p_z}{m} = \dot{z}$$

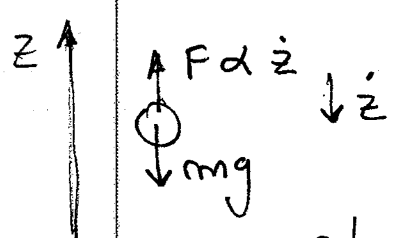
$$0 = -\dot{p}_x \qquad 0 = -\dot{p}_y \qquad mg = -\dot{p}_z$$

$m\ddot{x} = 0 \qquad m\ddot{y} = 0 \qquad m\ddot{z} = -mg$

$$\Rightarrow \dot{x} = \text{constant}$$
$$\dot{y} = \text{constant}$$
$$\ddot{z} = -g$$
$$\dot{z} = \dot{z}_0 - gt$$
$$z = z_0 + \dot{z}_0 t - \frac{1}{2} g t^2$$

which correspond to the known equations of motion for Projectile motion.

d) $\quad$ Net $F = -mg - km\dot{z}$ $\quad$ (Note $\dot{z} < 0$, $-km\dot{z}$ opposes motion)



$$-\cancel{m}g - k\cancel{m}\dot{z} = \cancel{m}\frac{d\dot{z}}{dt} \Rightarrow -dt = \frac{d\dot{z}}{(g+k\dot{z})}$$

$$\int_0^t -dt = \int_0^v \frac{d\dot{z}}{g+k\dot{z}} \Rightarrow -t = \ln(g+k\dot{z})\Big|_0^v \left(\frac{1}{k}\right)$$

$$\Rightarrow \boxed{v(t) = \frac{g}{k}\left(e^{-tk} - 1\right)}$$

Terminal velocity $t \to \infty$, $v_{term} = -g/k$

<u>Note</u>: at terminal velocity, Net Force $= 0$

Problem3

The rocket is an isolated system and to conserve linear momentum, it will have to move in the opposite than the ejected mass. We assume all motion is in the x direction and eliminate the vector notation. We know that the rate at which mass is ejected is constant, therefore $\mu = \frac{dm}{dt}$ and use the conservation of linear momentum before and after the rocket ejected a mass dm.

Initial momentum at time t: mv

NOTE both m and v are functions of time.

Final momentum at time t+dt: $(m-dm)(v+dv)+d\,m(v-u)$

NOTE: First term is rocket, second term is expelled gas, where (v-u) is the velocity of the gas with respect to the inertial reference system.

Conservation of linear momentum:

$$p(t) = p(t+dt)$$

$$mv = (m-dm)(v+dv)+dm(v-u)$$

$$m\frac{dv}{dt} = u\mu \quad \rightarrow \qquad dv = u\mu\frac{dt}{m(t)} \qquad \text{integrating} \qquad \int dv = \frac{u\mu}{M_0}\int\frac{dt}{(1-\frac{\mu t}{M_0})}$$

Gives $v(t) = -u\ln\left(1-\frac{\mu t}{M_0}\right)$

Integrating again & using that $\int \ln x\,dx = x\ln x - x$

$$x(t) = u\left[\left(\frac{M_0}{\mu}-t\right)\ln\left(1-\frac{\mu t}{M_0}\right)+1\right]$$

At $t = t_{1/2}$, the mass of the rocket has halved $\mu t_{1/2} = \frac{M_0}{2}$ and the distance travelled is

$$x(t_{1/2}) = ut_{1/2}(1-\ln 2)$$

Problem 4

a) The vertical displacement of the yo-yo is given by $x + a\theta$. The translational kinetic energy becomes $K_T = \frac{1}{2} m (\dot{x} + a\dot{\theta})^2$.

The rotational kinetic energy is $K_R = \frac{1}{2} I \dot{\theta}^2$

$\Rightarrow$ the total kinetic energy is

$$K = \frac{1}{2} m (\dot{x} + a\dot{\theta})^2 + \frac{1}{2} I \dot{\theta}^2$$

The potential energy also has 2 components:

gravitational $\quad \mathcal{U}_G = -mg(x + a\theta)$

elastic $\quad \mathcal{U}_e = \frac{1}{2} k x^2$

$\Rightarrow$ total potential energy

$$\mathcal{U} = -mg(x + a\theta) + \frac{1}{2} k x^2$$

$$\left[\mathcal{L} = K - \mathcal{U} = \frac{1}{2} m (\dot{x} + a\dot{\theta})^2 + \frac{1}{2} I \dot{\theta}^2 + mg(x + a\theta) - \frac{1}{2} k x^2 \right]$$

a) The Lagrange equations of motion for $q = x, \theta$ are:

(a) $$\frac{\partial L}{\partial x} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] = 0$$

$$m(\ddot{x} + a\ddot{\theta}) - mg + kx = 0 \qquad (1)$$

$$ma(\ddot{x} + a\ddot{\theta}) + I\ddot{\theta} - mga = 0 \qquad (2)$$

(b) From (2) $$\ddot{\theta} = \frac{ma(g - \ddot{x})}{ma^2 + I}$$

Substituting into Eq (1), we get a differential equation for x of the form
$$\ddot{x} + \omega^2 x = c$$

(1) $\Rightarrow$ $$m\ddot{x} + \frac{m^2a^2(g - \ddot{x})}{ma^2 + I} - mg + kx = 0$$

$$m\ddot{x}(ma^2 + I) + m^2a^2(g - \ddot{x}) - mg(ma^2 + I) - kx(ma^2 + I) = 0$$

$$m\ddot{x}I - mgI + kx(ma^2 + I) = 0$$

$$\ddot{x}\frac{Im}{ma^2 + I} + kx = \frac{Img}{(ma^2 + I)}$$

$$\left[\ddot{x} + \frac{k}{\frac{Im}{ma^2 + I}}x = g\right]$$

(c) $$\omega^2 = \frac{k}{\dfrac{Im}{ma^2+I}} = \frac{k}{m}\left(1+\frac{ma^2}{I}\right)$$

oscillation frequency $\omega = \sqrt{\dfrac{k}{m}\left(1+\dfrac{ma^2}{I}\right)}$

(d) If $ma^2 \ll I$, $\omega = \sqrt{\dfrac{k}{m}}$. Yo-yo behaves like point mass

Eq of motion is $\ddot{x} + \omega^2 x = g$ (3)

Solution is given by the sum of the complementary function (i.e. solution of (3) with right hand $= 0$) and the particular solution x_p

$x_p = C$ / $\frac{k}{m} C = g \Rightarrow$ $[x_p = g/\omega^2]$

$[x_c = A\cos(\omega t + \varphi)] \Rightarrow x(t) = A\cos(\omega t+\varphi) + g/\omega^2$

Applying initial conditions $x(0) = 0$, $\dot{x}(0) = 0$

we get
$$\left.\begin{aligned} x(0) &= A\cos\varphi \\ \dot{x}(0) &= -A\omega\sin\varphi = 0 \Rightarrow \varphi = 0 \end{aligned}\right\} A = -g/\omega^2$$

$$\therefore \boxed{x(t) = -\frac{g}{\omega^2}\cos\omega t + g/\omega^2}$$

Motion describes a point mass oscillating with $\omega = \sqrt{\frac{k}{m}}$ around the equilibrium position of the vertical spring $x = mg/k$

Problem 5

a) For rolling without slipping, the lengths traveled along the perimeters of disks A and B must be equal to the arc length traveled along the track C.

$$a\phi = b\beta = (a+2b)(\alpha-\phi)$$

$$a\phi = (a+2b)(\alpha-\phi)$$

$$\phi(2a+2b) = \alpha(a+2b) \Rightarrow \boxed{\phi = \frac{\alpha(a+2b)}{2(a+b)}}$$

$$\beta = \frac{a}{b}\phi \Rightarrow \boxed{\frac{\alpha a(a+2b)}{2b(a+b)} = \beta}$$

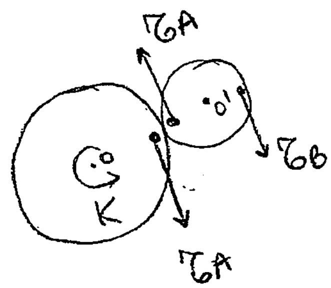
b) Angular velocity of B $\omega_B = \dot\beta - \dot\alpha + \dot\phi$

$$\omega_B = \frac{a(a+2b)}{2b(a+b)}\dot\alpha - \dot\alpha + \frac{(a+2b)}{2(a+b)}\dot\alpha =$$

$$= \frac{a(a+2b) - 2b(a+b) + b(a+2b)}{2b(a+b)}\dot\alpha =$$

$$= \frac{a^2 + \cancel{2ab} - \cancel{2ab} - \cancel{2b^2} + ba + \cancel{2b^2}}{2b(a+b)}\dot\alpha$$

$$= \frac{a\cancel{(a+b)}}{2b\cancel{(a+b)}}\dot\alpha \Rightarrow \boxed{\omega_B = \frac{a}{2b}\dot\alpha}$$



c) Torque equation for disc A about o

$$K - \tau_A a = I_A \ddot{\alpha} \qquad (1)$$

d) Torque equation for B about its center

$$-\tau_A b - \tau_B b = -I_B \dot{\omega}_B = -I_B \frac{a}{2b} \ddot{\alpha} \qquad (2)$$

e) Newton's law for B

$$\tau_A - \tau_B = M_B (a+b)(\ddot{\alpha} - \ddot{\phi})$$

$$= M_B (a+b)\left[\ddot{\alpha} - \ddot{\alpha}\,\frac{(a+2b)}{2(a+b)}\right] =$$

$$= M_B \ddot{\alpha}\,\cancel{(a+b)}\,\frac{\left[2(a+b) - (a+2b)\right]}{2\cancel{(a+b)}}$$

$$\tau_A - \tau_B = \frac{1}{2} M_B a \ddot{\alpha} \qquad (3)$$

$$(1) \Rightarrow \quad \tau_A = \frac{K}{a} - \frac{I_A}{a}\ddot{\alpha}$$

$$(2) \Rightarrow \quad \tau_B = \frac{I_B a}{2b^2}\ddot{\alpha} - \tau_A = \left(\frac{I_B a}{2b^2} + \frac{I_A}{a}\right)\ddot{\alpha} - \frac{K}{a}$$

$$(3) \Rightarrow \quad \frac{K}{a} - \frac{I_A}{a}\ddot{\alpha} - \frac{I_B a}{2b^2}\ddot{\alpha} - \frac{I_A}{a}\ddot{\alpha} + \frac{K}{a} = \frac{1}{2} M_B a \ddot{\alpha}$$

$$\frac{2K}{a} - \left(\frac{2 I_A}{a} + \frac{I_B a}{2b^2}\right)\ddot{\alpha} = \frac{1}{2} M_B a \ddot{\alpha}$$

$$K = \frac{\ddot{\alpha}}{4b^2}\left[4b^2 I_A + M_B a^2 b^2 + a^2 I_B\right]$$

Integrating

$$Kt = \frac{\dot{\alpha}}{4b^2}\left(4b^2 I_A + M_B a^2 b^2 + a^2 I_B\right)$$

With $\dot{\alpha} = \omega_A$ at $t = t_0$

$$\omega_A = \frac{4b^2 K t_0}{4b^2 I_A + M_B a^2 b^2 + a^2 I_B}$$

For $I_A = \frac{1}{2} M_A a^2 \qquad I_B = \frac{1}{2} M_B b^2$

$$\omega_A = \frac{4K t_0}{a^2 \left(2 M_A + \frac{3}{2} M_B\right)}$$

$$\omega_B = \frac{a}{2b}\omega_A \Rightarrow \boxed{\omega_B = \frac{2K t_0}{ab\left(2M_A + \frac{3}{2}M_B\right)}}$$