University of Illinois at Chicago
Department of Physics

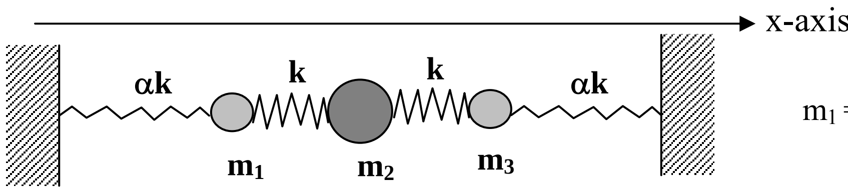
Classical Mechanics
Qualifying Examination

January 6, 2009
9.00 am – 12:00 pm

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted toward the exam's total score.

Problem 1

Long chain molecules sometimes have two types of bonds – strong bonds within small molecular units, and weak bonds between these molecular units. As a model for the normal mode vibrations in this situation, consider the longitudinal oscillations along the x-axis of the spring and ball system shown below, where the side walls are rigid. The springs obey Hooke's law with spring constants k and αk, and the masses m_i are related to each other as indicated. α and β are constants with $\alpha < 1$ and β arbitrary.



$m_1 = m_3 = m \qquad m_2 = \dfrac{m}{\beta}$

a) Give the Lagrangian that describes the small amplitude vibrations of this system parallel to the x-axis.

b) Obtain the equations for motion for harmonic oscillations, i.e., $x_i = a_i e^{i\omega t}$, where a_i are the amplitudes. Express your final equations in terms of the parameter $\lambda = \omega^2 \dfrac{m}{k}$.

c) Show that the dynamical matrix has the form $\begin{bmatrix} (1+\alpha) & -1 & 0 \\ -\beta & 2\beta & -\beta \\ 0 & -1 & (1+\alpha) \end{bmatrix}$.

Find the normal frequencies λ_o, λ_+, λ_- by diagonalizing the secular determinant for this motion.

[***Hint:*** *In your answer one of the frequencies, denoted here by* λ_o, *should depend only on* α.]

d) The eigenvector for λ_o has $a_2^0 = 0$. What is the relationship between a_1^0 and a_3^0 for this mode?

e) The eigenvectors for λ_+ and λ_- do not have $a_2 = 0$, but satisfy $a_2^{\pm} = a_1^{\pm} \big/ \left(1 - \lambda_{\pm}/2\beta\right)$.

In this case, use the equations of motion obtained in part (b) to prove that $a_1^{\pm} = a_3^{\pm}$.

[***Hint***: *To solve this, it is not necessary to use the explicit forms for* $\lambda_{\pm}(\alpha,\beta)$ *found in part (c).*]

For one of these two modes x_2 is in phase with x_1 and x_3, and for the other mode x_2 is out of phase with x_1 and x_3. Which mode has the higher frequency?

Problem 2

A pointlike mass m is undergoing a three-dimensional projectile motion in a uniform gravitational field g. Consider that the air resistance is negligible.

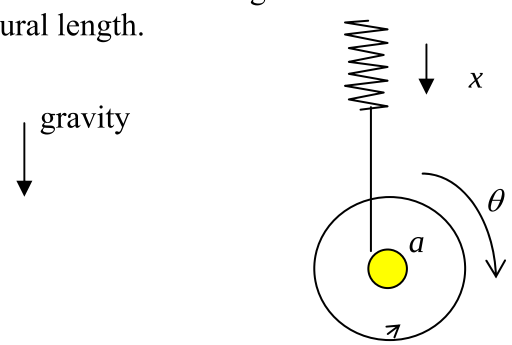
a) Write the Hamiltonian function for this problem $H = H(p_i, q_i)$, choosing as generalized coordinates the Cartesian coordinates (x,y,z) with the z axis pointing in the vertical direction.
b) Show that these equations lead to the known equations of motion for Projectile motion and that the Hamiltonian function corresponds to the Total Mechanical energy of the particle.
c) Once the particle reaches the highest point in its trajectory, a retarding force proportional to the velocity of the particle starts acting. Assume the proportionality constant to be known and given by k. Calculate the vertical velocity as a function of time for the descending particle, and find the terminal velocity.

Problem 3

Consider the one-dimensional motion of a rocket in outer space. The rocket is not subject to the influence of any external force, but rather moves by the reaction of ejecting mass at high velocities. At some arbitrary time t, the instantaneous total mass of the ship is $m(t)$, and its instantaneous velocity with respect to an inertial reference system is $v(t)$. During a time interval dt, a positive mass dm is ejected from the rocket engine with a speed $-u$ with respect to the rocket. Consider that the rate at which mass is ejected from the rocket is constant and given by μ. How far will the rocket have traveled once it lost half of its initial mass? Assume at the initial time the rocket's total mass is M_0, and it started from rest. Suppose the velocities involved are small enough to allow you to treat the problem with non-relativistic mechanics.

Problem 4

A yo-yo of mass m and rotational inertia I rolls down due to gravity. The end of its string is attached to a spring of negligible mass and spring constant k. The radius of the axle of the yo-yo is a, as indicated in the figure. Let x be the extension of the spring measured with respect to its natural length.



(a) Using the generalized coordinates x and θ indicated in the figure, write the Lagrange equations of motion for the yo-yo.
(b) Find the oscillation frequency of the spring while the yo-yo is rolling down.
(c) Consider a limit of a thin axle ($ma^2 << I$) and solve the differential equation for x you found in (b). Explain the motion that is described by the solution.

Problem 5

The figure illustrates two disks of radii a and b mounted inside a fixed circular track of radius c, such that $c = a + 2b$. The central disc A is mounted to a drive axle at point O. Disc B is sandwiched between disc A and track C and can roll without slipping when disc A is driven by an externally applied torque K through its drive axle. Initially, the system is at rest such that the dashed lines denoting the spatial orientation of discs A and B line up horizontally in the figure. K is then applied for a time t_0 causing disc A to rotate, such that at time t_0 the dashed line denoting its spatial orientation makes an angle α with the horizontal. Disc B rolls between the track and disc A, and its orientation is denoted by the dashed line making an angle β with the direction towards O. Let the moments of inertia of A and B be $I_A = \frac{1}{2}M_A a^2$ and $I_B = \frac{1}{2}M_B b^2$, respectively. The angular velocity of A is $\dot{\alpha}$ while that of B is $\dot{\beta} - \dot{\alpha} + \dot{\phi}$.

a) Taking into account that disc B is rolling without slipping, find expressions for the angles β and ϕ as a function of a, b, and α. HINT: For rolling without slipping, the lengths traveled along the perimeters of disks A and B must be equal to the arc length traveled along the track C.

b) Show that the angular velocity of disc B is equal to $\omega_B = \dfrac{a}{2b}\dot{\alpha}$

c) Calculate the final angular velocities of the two discs at $t = t_0$.