Electricity and Magnetism

Preliminary Exam

January 2009

1. Gauss law

A ball of radius R is uniformly charged with volume charge density equal to ρ. Use Gauss law and superposition principle to answer the following questions.

(a) Find the electric field $\boldsymbol{E}(\boldsymbol{r})$ *outside* the sphere as a function of the radius vector $\boldsymbol{r}$ drawn from the center of the sphere to the observation point.

(b) Find the electric field $\boldsymbol{E}(\boldsymbol{r})$ *inside* the sphere as a function of the radius vector $\boldsymbol{r}$ drawn from the center of the sphere to the observation point.

(c) Consider now the same sphere but now with an empty cavity. The center of the cavity is at distance a from the center of the sphere. The radius of the cavity is b, such that $a+b<R$. Find the electric field $\boldsymbol{E}$ inside the cavity.

2. Multipole expansion

The surface of a sphere of radius R is charged with surface charge density varying according to $\sigma = \sigma_0 \cos\theta$. Use the multipole expansion to answer the following questions.

(a) Find the potential Φ inside and outside the sphere as a function of r and θ.

(b) Find the electric field $\boldsymbol{E}$ inside the sphere.

3. Electrostatics of medium

A spherical capacitor consists of two concentric conducting spheres of radii a and b. The capacitor is filled with dielectric material whose dielectric constant varies according to:

$$\varepsilon(r) = \begin{cases} \varepsilon_1 = \text{const}, & \text{for } a < r < c \\ \varepsilon_2 = \text{const}, & \text{for } c < r < b \end{cases}$$

The charge on the *inner* conducting sphere is Q.

(a) Find the electric field inside the capacitor as a function of r.

(b) Find the capacitance of this capacitor.

(c) Find the density of the bound charge on the boundary between the dielectric layers at $r = c$.

4. Magnetostatics of medium

A very long circular solenoid is made out of a wire with n turns per unit length. The radius of the cylinder is a and is negligible compared to its length l. The interior of the cylinder is filled with material such that the linear magnetic permeability varies with the distance r from its axis according to:

$$\mu(r) = \begin{cases} \mu_1 = \text{const}, & \text{for } 0 < r < b \\ \mu_2 = \text{const}, & \text{for } b < r < a \end{cases}$$

The current passing through the wire is equal to I.

(a) Find the magnetic field $\boldsymbol{B}$ inside the solenoid as a function of r.

(b) Find the inductance L of such a solenoid.

5. Radiation

A thin rod of length L is charged uniformly with density λ per unit length. The rod is rotated with angular velocity ω around an axis passing through one of its ends perpendicular to the rod.

(a) Find the magnetic dipole moment $\boldsymbol{m}$ of the rod.

(b) Find the electric dipole $\boldsymbol{p}$ moment of the rod.

(c) Find the total radiation energy emitted by the rotating rod per unit time.

Equations

$$\boldsymbol{\nabla D} = \rho; \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -d\boldsymbol{B}/dt; \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} + d\boldsymbol{D}/dt; \quad \boldsymbol{\nabla B} = 0;$$

$$\boldsymbol{D} = \varepsilon\varepsilon_0 \boldsymbol{E}; \quad \boldsymbol{B} = \mu\mu_0 \boldsymbol{H};$$

$$\boldsymbol{p} = \int d^3x\, \rho \boldsymbol{r}; \quad \boldsymbol{m} = \frac{1}{2}\int d^3x\, \boldsymbol{r} \times \boldsymbol{J};$$

$$\frac{d\mathcal{E}}{dt} = \frac{|\boldsymbol{p}_\omega|^2\omega^4}{12\pi\varepsilon_0 c^3}; \quad \frac{d\mathcal{E}}{dt} = \frac{\mu_0|\boldsymbol{m}_\omega|^2\omega^4}{12\pi c^3};$$

$$\Phi = \sum_l (A_l r^l + B_l/r^{l+1}) P_l(\cos\theta);$$

$$P_0 = 1; \quad P_1 = x; \quad P_2 = \frac{1}{2}(3x^2 - 1).$$