Quantum Solutions

1. (a) Since ΔE is time-independent, we can calculate it at time t_1. We have

$$\langle H \rangle = \frac{E_1 + E_2}{2}, \quad \langle H^2 \rangle = \frac{E_1^2 + E_2^2}{2},$$

which gives

$$\Delta E \equiv \sqrt{\langle H^2 \rangle - \langle H \rangle^2} = \frac{|E_1 - E_2|}{2}\,.$$

(In the above, we used the fact that $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal since they are eigenstates corresponding to distinct eigenvalues of the Hamiltonian operator $H = H^\dagger$.) At any later time t, the state of the system is given by

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}(\mathrm{e}^{-iE_1(t-t_1)/\hbar}\,|\psi_1\rangle + \mathrm{e}^{-iE_2(t-t_1)/\hbar}\,|\psi_2\rangle)\,.$$

This is orthogonal to $|\Psi(t_1)\rangle$ if and only if $\mathrm{e}^{i(E_1-E_2)(t-t_1)/\hbar} = -1$. The smallest $t > t_1$ at which this occurs is $t_2 = t_1 + \pi\hbar/|E_1 - E_2|$. Thus,

$$\Delta t = \frac{\pi\hbar}{|E_1 - E_2|} \quad \text{and} \quad (\Delta E)(\Delta t) = \pi\hbar/2 > \hbar\,.$$

(b) The energy levels of the Coulomb potential are

$$E_{n,\ell} = -\frac{Me^4}{2\,\hbar\,n^2}\,.$$

Thus,

$$(\Delta E) = \frac{3Me^4}{16\,\hbar^2} \quad \text{and} \quad (\Delta t) = \frac{8\pi\hbar^3}{3Me^4}\,.$$

Since the Coulomb Hamiltonian H is rotationally invariant, we have $[H, L^2] = [H, L_z] = \hat{0}$, so that $\langle L^2 \rangle$ and $\langle L_z \rangle$ are time-independent. Moreover, since $L^2\,|\psi_{n,\ell,m}\rangle = \ell(\ell+1)\hbar^2$ and $L_z\,|\psi_{n,\ell,m}\rangle = m\hbar$, we have

$$\langle L^2 \rangle = \hbar^2 \quad \text{and} \quad \langle L_z \rangle = 0\,.$$

(Note that, for all t, $|\Psi(t)\rangle$ is an eigenstate of L_z but not of L^2.)

2. (a) First note that after the second measurement, the system must be in an eigenstate of A (by the projection postulate). But this state is $|\psi\rangle$ by assumption. Thus, $|\psi\rangle$ is an eigenstate of A, say with eigenvalue λ.

Now assume that A and B commute. (We want to show that this leads to a contradiction.) We then find

$$AB\,|\psi\rangle = BA\,|\psi\rangle = \lambda B\,|\psi\rangle\ .$$

That is, $B\,|\psi\rangle$ is an eigenstate of A, also with eigenvalue λ. But since A is non-degenerate, any two eigenstates of A with the same eigenvalue must be colinear. Thus, $B\,|\psi\rangle = b\,|\psi\rangle$ for some

(real) number b. In other words, $|\psi\rangle$ is an eigenstate of B as well. But if this were true, then the initial measurement of B would not change the state of the system (again by the projection postulate) — a contradiction.

(b) Consider a two-dimensional Hilbert space with orthonormal basis $\{\,|1\rangle\,,|2\,\rangle\,\}$. Let $|\psi\rangle = |1\rangle$, $A = |1\rangle\langle 1|$, and $B = |1\rangle\langle 2| + |2\rangle\langle 1|$. The eigenvalues of A are $\lambda = 1$ and $\lambda = 0$, with corresponding eigenvectors $|1\rangle$ and $|2\rangle$ respectively. The eigenvalues of B are $\lambda = \pm 1$, with corresponding eigenvectors $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$. If a measurement of B is performed on $|\psi\rangle$ we will obtain one of the two eigenvalues, each with equal probability (=1/2 by Born's Rule). Lets say we get $\lambda = +1$. The state of the system immediately following the measurement will be the corresponding eigenstate $|\psi_+\rangle$. Upon the ensuing measurement of A, we will again obtain either of the two eigenvalues with equal probability. Lets say we get $\lambda = 1$. Then the state of the system after the second measurement will be $|1\rangle = |\psi\rangle$, as desired. The probability of this scenario occuring (that is, of obtaining this particular pair of eigenvalues) is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

3. (a) Assume that $H_0|\psi_m^{(0)}\rangle = E_m^{(0)}|\psi_m^{(0)}\rangle$, where the $|\psi_m^{(0)}\rangle$'s are a complete, orthonormal set of eigenstates of H_0. The first-order corrections to $E_n^{(0)}$ and $|\psi_n^{(0)}\rangle$ in non-degerate perturbation theory are given by

$$E_n^{(1)} = \langle\psi_n^{(0)}|H_1|\psi_n^{(0)}\rangle \quad \text{and} \quad |\psi_n^{(1)}\rangle = \sum_{m\neq n} \frac{\langle\psi_m^{(0)}|H_1|\psi_n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}}|\psi_m^{(0)}\rangle\,.$$

Since $|\psi_n^{(1)}\rangle$ is the zero vector, we have by the linear independence of the $|\psi_m^{(0)}\rangle$'s that

$$\langle\psi_m^{(0)}|H_1|\psi_n^{(0)}\rangle = 0 \quad \text{for all} \quad m \neq n\,.$$

That is, $H_1|\psi_n^{(0)}\rangle = \lambda|\psi_n^{(0)}\rangle$ for some complex number λ, and thus $E_n^{(1)} = \lambda$. But since $E_n^{(1)} = 0$, we have that $H_1|\psi_n^{(0)}\rangle$ is the zero vector. From this we see that $H|\psi_m^{(0)}\rangle = E_n^{(0)}|\psi_n^{(0)}\rangle$. That is, $|\psi_n^{(0)}\rangle$ is an *exact* eigenvector of the *full* Hamiltonian H (with eigenvalue $E_n^{(0)}$), and hence there are no corrections to *all* orders in λ.

(b) We have from (a) that $V_1(\hat{x})|\psi_n^{(0)}\rangle$ is the zero vector. In the position representation we have (using the completeness of position eigenstates)

$$V_1(\hat{x})|\psi_n^{(0)}\rangle = \int_{-\infty}^{+\infty} V_1(\hat{x})|x\rangle\langle x|\psi_n^{(0)}\rangle\,dx = \int_{-\infty}^{+\infty} \psi_n^{(0)}(x)V_1(x)|x\rangle\,dx\,.$$

Since this is the zero vector, we have that $V_1(x)\psi_n^{(0)}(x) = 0$ for all x. Hence $V_1(x) = 0$ for all x (except possibly at the nodes of $\psi_n^{(0)}(x)$ which are a set of measure zero), which implies that $V_1(\hat{x}) = \hat{0}$.

4. (a) Here we have $V_{\text{ext}}(\vec{x}) = A\,(x^2 + y^2 + z^2)$, where $A > 0$. The energy eigenvalues for a single (spinless) particle of mass M in this potential are given by

$$E_{n_x,n_y,n_z} = (n_x + n_y + n_z + 3/2)\,\hbar\,\omega$$

where $n_x, n_y, n_z = 0, 1, 2, \ldots$, and (for simplicity) we have chosen $A = \frac{1}{2}M\omega^2$. If we define $N = n_x + n_y + n_z$, then any two states of the single-particle system with the same N have the

same energy. It is easy to see that the degeneracy of the level corresponding to any fixed N is simply $(N+1)(N+2)/2$. (It is easy to work out the corresponding n_i's.)

For our multi-particle system, by the Pauli exclusion principle, each of the above states can accomodate exactly two of the identical spin 1/2 fermions — one with "spin up", and one with "spin down". To find the degeneracy of the lowest-lying level of the 15 particles, we imagine filling these single-particle levels from the "bottom up". The $N = 0$ level can accomodate 2 fermions, and the $N = 1$ level can hold 6 fermions. This leaves 11 fermions. The $N = 2$ level can accomodate 12 fermions. Thus there are 12 distinct ways of filling this level with the remaining 11 fermions. (We just have to decide which one of the 12 states we *don't* occupy.) Thus, the degeneracy of the ground level of the 15 identical spin 1/2 fermions is 12.

(b) The energy eigenvalues for a single (spinless) particle of mass M in a cubical box of side-length "a" are given by

$$E_{n_x,n_y,n_z} = \frac{\hbar^2\pi^2}{2Ma^2}(n_x^2 + n_y^2 + n_z^2)\,,$$

where $n_x, n_y, n_z = 1, 2, \ldots$. If we define $N = n_x^2 + n_y^2 + n_z^2$, then any two states of the single-particle system with the same N have the same energy. The ground state of the single-particle system has $N = 3$ and is non-degenerate. The first excited state has $N = 6$ and is 3-fold degenerate. The second excited state has $N = 9$ and is also 3-fold degenerate. (It is easy to work out the corresponding n_i's.)

For a multi-particle system of identical bosons, each of the above states can accomodate as many particles as we wish. (There is no exclusion principle.) However, since the bosons are identical, we must still (as for fermions) be careful not to overcount multi-particle states. A two-particle state is labelled by an *un-ordered* pair of single-particle states (which get symmetrized). Lets work our way up the two-particle spectrum. The ground level of the two bosons has them both in the $N = 3$ single-particle level, and is non-degenerate. The first excited level has one particle with $N = 3$ and the other with $N = 6$. It's degeneracy is $1 \times 3 = 3$. The second excited level has either one particle with $N = 3$ and the other with $N = 9$, or both with $N = 6$. It's degeneracy is $1 \times 3 + 3 \times 3 = 12$.

5. (a) It is simplest to work with the reduced radial Schrödinger equation

$$\hat{H}_\ell u_{n,\ell}(r) \equiv \big[-\frac{\hbar^2}{2M}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)\hbar^2}{2M} + V(r)\big] u_{n,\ell}(r) = E_{n,\ell}\, u_{n,\ell}(r)\,,$$

where $u_{n,\ell}(r) = rR_{n,\ell}(r)$ is the reduced radial wavefunction and $R_{n,\ell}(r)$ is the full radial wavefunction. The normalization integral for $u_{n,\ell}(r)$ is

$$\int_0^\infty |u_{n,\ell}(r)|^2 dr = 1\,.$$

Our trial ground state wavefunction in its reduced form now becomes $u(r) = Cr\mathrm{e}^{-\alpha r}$, where C is a normalization constant. (We can actually do this problem without having to normalize $u(r)$, but I will do the integral anyway below.) Since in our case $V(r) \to 0$ as $r \to \infty$, we have by the variational principle that if for any fixed β there exists a value of α such that $\langle \hat{H}_0 \rangle_u < 0$, then there is a bound state of the system for that value of β. Using the integral

$$\int_0^\infty r^n \mathrm{e}^{-ar} dr = \frac{n!}{a^3}$$

(for n a non-negative integer and $a > 0$), we easily find that $C = 2\alpha^{3/2}$ and

$$\langle \hat{H}_0 \rangle_u = \frac{\hbar^2 \alpha^2}{2M} - \frac{8A\alpha^3}{(2\alpha + \beta)^3} \, .$$

As β decreases, the second term approaches the negative constant $-A$. But the first term (which is the positive kinetic energy term) can be made as small as we want by decreasing α. Therefore, for "small enough" β we can always find a "large enough" α so that $\langle \hat{H}_0 \rangle_u < 0$.

(b) It is difficult to answer this question "rigorously", which is why I only asked the students what they "expect". What I am looking for is the following type of response:

"I do not expect there to be a bound state for all β since the expectation value of the potential energy for *any* trial wavefunction will go to zero very rapidly as $\beta \to \infty$. And since the potential is getting "narrower" in this limit, by the uncertainty principle I don't expect that the corresponding expectation value of the kinetic energy can become small enough so that the sum remains negative."

(Note that even though for any fixed α in our trial wavefunction in (a) there exists a large enough β such that $\langle \hat{H}_0 \rangle_u > 0$, we can't conclude from this that there are no bound states for large β since $\langle \hat{H}_0 \rangle_u$ is in general only an *upper bound* for the exact ground state energy. That is, the exact ground state wavefunction may not be of the form from (a).)