Quantum Problems

1. Consider a quantum system whose state at time t_1 is given by $|\Psi(t_1)\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$, where $|\psi_1\rangle$ and $|\psi_2\rangle$ are energy eigenstates with eigenvalues E_1 and E_2 respectively. ($E_1 \neq E_2$.)

(a) Calculate the uncertainty ΔE of the system, as well as the first time $t_2 > t_1$ at which $|\Psi(t_2)\rangle$ becomes orthogonal to $|\Psi(t_1)\rangle$. Show that $(\Delta E)(\Delta t) \geq \hbar$, where $\Delta t = t_2 - t_1$.

(b) Assume that the above system consists of a particle of mass M moving in the Coulomb potential $V(r) = -e^2/r$ in three spatial dimensions, and that $|\psi_1\rangle = |\psi_{1,0,0}\rangle$ and $|\psi_2\rangle = |\psi_{2,1,0}\rangle$, where $|\psi_{n,\ell,m}\rangle$ is the energy eigenstate with principal quantum number n, angular momentum quantum number ℓ, and magnetic quantum number m. Find ΔE and Δt in (a) as a function of M, e, and $\hbar$. Also calculate the time-evolving expectation values $\langle L^2\rangle(t)$ and $\langle L_z\rangle(t)$ for all $t \geq t_1$, where L^2 and L_z are the usual angular momentum operators.

2. Two measurements are made in rapid succession on a quantum system originally in the state $|\psi\rangle$. The first measurement is of an observable B, and the second is of a non-degenerate observable A. Assume that the first measurement changes the state of the system, and that immediately after the second measurement the system is again in the state $|\psi\rangle$.

(a) Prove that $[A, B] \neq \hat{0}$.

(b) Show, via an explicit example, that the physical situation described above can actually occur. (Suggestion: Try this for a two-dimensional Hilbert space.) What is the probability of your particular scenario occuring?

3. Consider a quantum system with Hamiltonian operator $H = H_0 + \lambda H_1$, and let $|\psi_n^{(0)}\rangle$ be a particular non-degenerate eigenstate of H_0 with corresponding eigenvalue $E_n^{(0)}$. Assume $\lambda \ll 1$.

(a) Show that if the first-order corrections in λ to *both* $|\psi_n^{(0)}\rangle$ and $E_n^{(0)}$ vanish, then all higher order corrections to both vanish as well. (Hint: Prove that if the first-order corrections vanish, then $H_1|\psi_n^{(0)}\rangle$ is the zero vector. In this problem, as is standard, we have chosen the first-order correction to $|\psi_n^{(0)}\rangle$ to be orthogonal to $|\psi_n^{(0)}\rangle$.)

(b) In the special case of single particle motion in one dimension with $H_0 = \hat{p}^2/2M + V_0(\hat{x})$ and $H_1 = V_1(\hat{x})$, show that if the first-order corrections in (a) vanish, then $H_1 = \hat{0}$.

4. Calculate the degree of degeneracy of the indicated energy level for the following multi-particle systems in three spatial dimensions.

(a) The ground level of 19 identical spin 1/2 fermions moving in an external isotropic harmonic oscillator potential.

(b) The second excited level of 2 identical spinless bosons confined inside a cubical box.

5. Consider a particle of mass M moving in the three-dimensional spherically symmetric potential $V(r) = -A\,\mathrm{e}^{-\beta r}$, where $A, \beta > 0$.

(a) Show that for "small enough" values of β, the system possesses at least one bound state. (Hint: Use the variational method with trial ground state wavefunction $\psi(r) = N\mathrm{e}^{-\alpha r}$, where $\alpha > 0$ and N is a normalization constant. No minimization with respect to α is necessary.)

(b) Do you expect the system to have a bound state for *all* values of β? Why or why not?