1. Consider a ring of uniform charge density λ and radius R that lies within the xy-plane. The origin of the coordinate systems is located at the center of the ring.

a) Give the potential at the point $\vec{P} = (\rho_0, \varphi, z)$ in terms of $\lambda, R, \rho_0, \varphi,$ and z.

b) We next put a conducting plane into the $z = d$ plane. The potential of the conducting plane is fixed at $V = 0$. Compute the total potential at a point $\vec{P} = (\rho_0, \varphi, z)$.

c) Give an explicit form of the induced charge density at $\vec{P} = (0, 0, d)$? Your final answer should contain no integrals or derivatives.

Solution:

a) Give the potential at the point $\vec{P} = (\rho_0, \varphi, z)$ in terms of $\lambda, R, \rho_0, \varphi,$ and z.

Let $\vec{r} = R(\cos\varphi, \sin\varphi, 0)$ be a point along the ring, then

$$\begin{aligned}
\Phi(\vec{P}) &= \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{\left|\vec{P}-\vec{r}\right|} = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda dl}{\left|\vec{P}-\vec{r}\right|} = \frac{\lambda R}{4\pi\varepsilon_0}\int_0^{2\pi} \frac{d\varphi}{\sqrt{(R\cos\varphi-\rho_0)^2+(R\sin\varphi)^2+z^2}} \\
&= \frac{\lambda R}{4\pi\varepsilon_0}\int_0^{2\pi} \frac{d\varphi}{\sqrt{R^2+\rho_0^2+z^2-2\rho_0 R\cos\varphi}} \\
&= \frac{\lambda R}{4\pi\varepsilon_0}\frac{1}{\sqrt{R^2+\rho_0^2+z^2-2\rho_0 R}}K\left[\frac{-2\rho_0 R}{R^2+\rho_0^2+z^2-2\rho_0 R}\right]
\end{aligned}$$

where K is the complete elliptic integral of the first kind.

b) We next put a conducting plane into the $z = d$ plane. The potential of the conducting plane is fixed at $V = 0$. Compute the total potential at a point $\vec{P} = (\rho_0, \varphi, z)$.

Using the method of images, the image charge is a ring of charge density $-\lambda$ and radius R that is parallel to the conducting plane at a distance of $2d$ from the first ring. The resulting total potential is then

$$\begin{aligned}
\Phi_{tot}(\vec{P}) &= \frac{\lambda R}{4\pi\varepsilon_0}\frac{1}{\sqrt{R^2+\rho_0^2+z^2-2\rho_0 R}}K\left[\frac{-2\rho_0 R}{R^2+\rho_0^2+z^2-2\rho_0 R}\right] \\
&\quad - \frac{\lambda R}{4\pi\varepsilon_0}\frac{1}{\sqrt{R^2+\rho_0^2+(2d-z)^2-2\rho_0 R}}K\left[\frac{-2\rho_0 R}{R^2+\rho_0^2+(2d-z)^2-2\rho_0 R}\right]
\end{aligned}$$

c) Give an explicit form of the induced charge density at $\vec{P} = (0, 0, d)$? Your final answer should contain no integrals or derivatives.

The charge density follows from

$$\sigma = -\varepsilon_0 \frac{\partial \Phi_{tot}}{\partial(-z)}\bigg|_{z=d}$$

Note that at $\vec{P} = (0, 0, d)$, we can compute the potential simply via

$$\Phi(\vec{P}) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{\left|\vec{P}-\vec{r}\right|} = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda dl}{\left|\vec{P}-\vec{r}\right|} = \frac{\lambda R}{4\pi\varepsilon_0}\int_0^{2\pi} \frac{d\varphi}{\sqrt{R^2+z^2}} = \frac{\lambda R}{2\varepsilon_0\sqrt{R^2+z^2}}$$

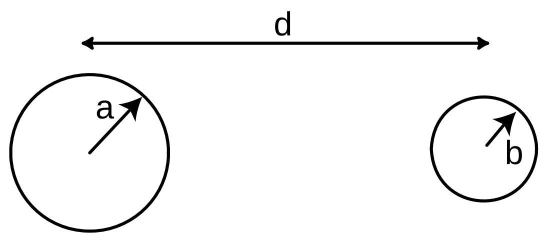
and hence

$$\Phi_{tot}(\vec{P}) = \frac{\lambda R}{2\varepsilon_0}\left[\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{\sqrt{R^2+(2d-z)^2}}\right]$$

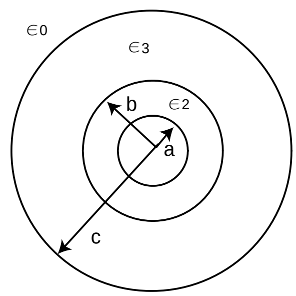
and thus

$$\sigma = -\varepsilon_0 \frac{\partial \Phi_{tot}}{\partial(-z)}\Big|_{z=d} = \varepsilon_0 \frac{\lambda R}{2\varepsilon_0}\left(-\frac{1}{2}\right)\left[\frac{2z}{(R^2+z^2)^{3/2}} - \frac{2(z-2d)}{\left[R^2+(2d-z)^2\right]^{3/2}}\right]_{z=d} = -\frac{d\lambda R}{(R^2+d^2)^{3/2}}$$

2. a) Using Gauss' law, compute the capacitance per unit length of two infinitely long cylindrical conductors of radii a and b that are parallel and separated by a distance $d \gg a, b$, as shown in the figure below



b) Consider next two infinitely long concentric cylinders, as shown in the figure below. The inner cylinder of radius a is a conductor with linear charge density $\lambda_1 > 0$. The second cylinder with inner radius b and outer radius c consists of a material with permitivity ε_3 and is uniformly charged with line charge density $\lambda_3 < 0$ ($\lambda_1 > |\lambda_3|$). The space between the two cylinders (i.e., $a < r < b$) is filled with a medium of permitivity ε_2. The medium outside the outer cylinder possesses the permitivity ε_0. Compute the potential difference between a point at $|\vec{r}| = 2c$ and the center of the inner cylinder.



.

Solution:

a) Using Gauss' law, compute the capacitance per unit length of two infinitely long cylindrical conductors of radii a and b that are parallel and separated by a distance $d \gg a, b$.

Using Gauss' law, we have outside the cylinders

$$\int \vec{E} \cdot d\vec{a} = 2\pi r l E = \frac{\lambda l}{\varepsilon_0} \Rightarrow E = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}$$

Moreover, the electric fields of the two cylinders are pointing in the same direction on the line connnecting them. I thus obtain

$$\Delta\Phi = -\int \vec{E}_{tot} \cdot d\vec{r} = -\frac{\lambda}{2\pi\varepsilon_0}\left[\int_a^{d-b} \frac{1}{r}dr + \int_b^{d-a} \frac{1}{r}dr\right] = -\frac{\lambda}{2\pi\varepsilon_0} \ln\left[\frac{(d-b)(d-a)}{ab}\right]$$

and thus

$$C = \frac{Q}{|\Delta\Phi|} = \frac{\lambda l}{\frac{\lambda}{2\pi\varepsilon_0}\ln\left[\frac{(d-b)(d-a)}{ab}\right]} \Rightarrow \frac{C}{l} = \frac{2\pi\varepsilon_0}{\ln\left[\frac{(d-b)(d-a)}{ab}\right]}$$

b) Consider next two infinitely long concentric cylinders. The inner cylinder of radius a is a conductor with linear charge density $\lambda_1 > 0$. The second cylinder with inner radius b and outer radius c consists of a material with permitivity ε_3 and is uniformly charged with line charge density $\lambda_3 < 0$ ($\lambda_1 > |\lambda_3|$). The space between the two cylinders (i.e., $a < r < b$) is filled with a medium of permitivity ε_2. The medium outside the outer cylinder possesses the permitivity ε_0. Compute the potential difference between a point at $|\vec{r}| = 2c$ and the center of the inner cylinder.

We need to compute the electric field in the different regions of the problem.

i) $r < a$. We have $E = 0$ inside the inner cylinder is zero, since it is a conductor.

ii) $a < r < b$. We use

$$\int \vec{D} \cdot d\vec{a} = \lambda_1 l \Rightarrow 2\pi r l D = \lambda_1 l \Rightarrow D = \frac{1}{2\pi}\frac{\lambda_1}{r}$$

and thus

$$E = \frac{D}{\varepsilon_2} = \frac{1}{2\pi\varepsilon_2}\frac{\lambda_1}{r}$$

Note that since $\lambda_1 > 0$ the electric field points radially outwards.

iii) $b < r < c$. Note that the material in this region is insulating and uniformly charged. The volume charge density is

$$\rho = \frac{\lambda_3}{\pi(c^2 - b^2)}$$

and thus

$$\int \vec{D} \cdot d\vec{a} = \left(\lambda_1 + \lambda_3\frac{(r^2 - b^2)}{(c^2 - b^2)}\right)l \Rightarrow 2\pi r l D = \left(\lambda_1 + \lambda_3\frac{(r^2 - b^2)}{(c^2 - b^2)}\right)l \Rightarrow D = \frac{1}{2\pi}\frac{1}{r}\left(\lambda_1 + \lambda_3\frac{(r^2 - b^2)}{(c^2 - b^2)}\right)$$

and thus

$$E = \frac{D}{\varepsilon_3} = \frac{1}{2\pi\varepsilon_3}\frac{1}{r}\left(\lambda_1 + \lambda_3\frac{(r^2 - b^2)}{(c^2 - b^2)}\right)$$

iv) $c < r$. Here we have

$$E = \frac{1}{2\pi\varepsilon_0}\frac{\lambda_1 + \lambda_3}{r}$$

We can now compute the potential difference

$$\Delta\Phi = -\int_{2c}^{0} \vec{E}_{tot} \cdot d\vec{r} = -\int_{2c}^{c} E\,dr - \int_{c}^{b} E\,dr - \int_{b}^{a} E\,dr - \int_{a}^{0} E\,dr$$

where

$$-\int_{2c}^{c} E\,dr = -\int_{2c}^{c} \frac{1}{2\pi\varepsilon_0}\frac{\lambda_1 + \lambda_3}{r}dr = \frac{\lambda_1 + \lambda_3}{2\pi\varepsilon_0}\ln 2$$

and

$$-\int_c^b E\,dr = -\int_c^b \frac{1}{2\pi\varepsilon_3}\frac{1}{r}\left(\lambda_1 + \lambda_3\frac{(r^2-b^2)}{(c^2-b^2)}\right)dr = -\int_c^b \frac{1}{2\pi\varepsilon_3}\frac{1}{r}\left(\lambda_1 - \lambda_3\frac{b^2}{(c^2-b^2)}\right) - \int_c^b \frac{\lambda_3}{2\pi\varepsilon_3}\frac{r}{(c^2-b^2)}dr$$

$$= \frac{1}{2\pi\varepsilon_3}\left(\lambda_1 - \lambda_3\frac{b^2}{(c^2-b^2)}\right)\ln\frac{c}{b} + \frac{\lambda_3}{4\pi\varepsilon_3}$$

and

$$-\int_b^a E\,dr = -\int_b^a \frac{1}{2\pi\varepsilon_2}\frac{\lambda_1}{r}dr = \frac{\lambda_1}{2\pi\varepsilon_2}\ln\frac{b}{a}$$

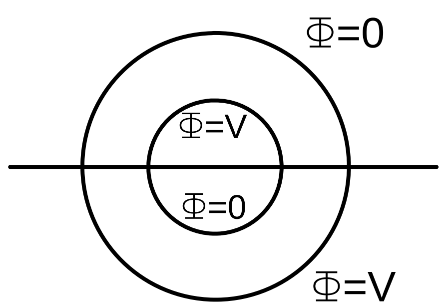
and

$$-\int_a^0 E\,dr = 0$$

Thus I obtain

$$\Delta\Phi = \frac{\lambda_1+\lambda_3}{2\pi\varepsilon_0}\ln 2 + \frac{1}{2\pi\varepsilon_3}\left[\left(\lambda_1 - \lambda_3\frac{b^2}{(c^2-b^2)}\right)\ln\frac{c}{b} + \frac{\lambda_3}{2}\right] + \frac{\lambda_1}{2\pi\varepsilon_2}\ln\frac{b}{a}$$

3. Two concentric spheres have radii a and b ($b > a$) and each is divided into two hemispheres by the same horizontal plane, as shown below. The potential of the upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere is kept at V. The other hemispheres are at zero potential.



a) Derive an explicit form of the boundary conditions, using a series expansion of the potential in Legendre Polynomials.

b) Derive an expression for the coefficients (to all order) in the series expansion of the potential in the region $a < r < b$. Give the explicit form of the coefficients up to $l = 2$.

c) Which coefficients in the series vanish for $r < a$ and which vanish for $r > b$? Why?

Solution:

a) Derive an explicit form of the boundary conditions, using a series expansion of the potential in Legendre Polynomials.

The problem possesses an azimuthal symmetry, hence

$$
\begin{aligned}
\Phi(r,\theta) &= \sum_n [A_n r^n + B_n r^{-(n+1)}] P_n(\cos\theta) \\
&\Rightarrow \int_{-1}^{1} dx \Phi(r,x) P_l(x) = \sum_n [A_n r^n + B_n r^{-(n+1)}] \int_{-1}^{1} dx P_n(x) P_l(x) \\
&= \sum_n [A_n r^n + B_n r^{-(n+1)}] \frac{2\delta_{l,n}}{2l+1} = \frac{2}{2l+1} [A_l a^l + B_l a^{-(l+1)}]
\end{aligned}
$$

And thus we have as the boundary conditions

$$
\begin{aligned}
\int_{-1}^{1} dx\, \Phi(a,x) P_l(x) &= \frac{2}{2l+1} [A_l a^l + B_l a^{-(l+1)}] \\
\int_{-1}^{1} dx\, \Phi(b,x) P_l(x) &= \frac{2}{2l+1} [A_l b^l + B_l b^{-(l+1)}]
\end{aligned}
$$

b) Derive an explicit expression for the coefficients (to all order) in the series expansion of the potential in the region $a < r < b$.

In order to derive the coefficients, I note that

$$
\int_{-1}^{1} dx\, \Phi(a,x) P_l(x) = V \int_{0}^{1} dx\, P_l(x) = V \begin{cases} 0 & \text{for even } l \\ 1 & \text{for } l = 0 \\ (-1)^{\frac{l-1}{2}} \frac{(l+1)(l-1)!}{2^{l+1}\left[\left(\frac{l+1}{2}\right)!\right]^2} & \text{for odd } l \end{cases} \equiv VI_l
$$

and similarly

$$
\int_{-1}^{1} dx\, \Phi(b,x) P_l(x) = V \int_{-1}^{0} dx P_l(x) = V(-1)^l \int_{0}^{1} dx\, P_l(x)
$$

hence I obtain

$$
\begin{aligned}
VI_l &= \frac{2}{2l+1} [A_l a^l + B_l a^{-(l+1)}] \\
V(-1)^l I_l &= \frac{2}{2l+1} [A_l b^l + B_l b^{-(l+1)}]
\end{aligned}
$$

and we can simply solve from the first equation

$$
A_l = \frac{1}{a^l} \left[\frac{2l+1}{2} VI_l - B_l a^{-(l+1)} \right]
$$

and inserted into the second equation

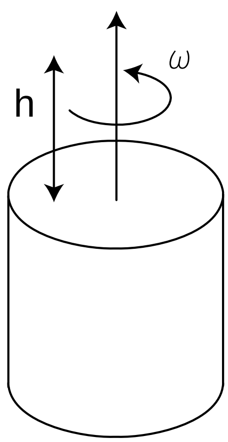
$$
\begin{aligned}
V(-1)^l I_l &= \frac{2}{2l+1} \left[\frac{b^l}{a^l} \frac{2l+1}{2} VI_l + B_l b^{-(l+1)} - \frac{b^l}{a^l} B_l a^{-(l+1)} \right] \\
\left((-1)^l - \frac{b^l}{a^l} \right) VI_l &= \frac{2}{2l+1} B_l a^{-(l+1)} \left[\left(\frac{a}{b} \right)^{(l+1)} - \left(\frac{b}{a} \right)^{l} \right] \\
\left[1 - \left(-\frac{a}{b} \right)^{l} \right] VI_l &= \frac{2}{2l+1} B_l a^{-(l+1)} \left[1 - \left(\frac{a}{b} \right)^{(2l+1)} \right] \\
B_l &= \frac{2l+1}{2} \left[1 - \left(-\frac{a}{b} \right)^{l} \right] VI_l a^{(l+1)} \left[1 - \left(\frac{a}{b} \right)^{(2l+1)} \right]^{-1}
\end{aligned}
$$

and hence

$$
\begin{aligned}
A_l &= \frac{1}{a^l}\left[\frac{2l+1}{2}VI_l - \frac{2l+1}{2}\left[1-\left(-\frac{a}{b}\right)^l\right]VI_l a^{(l+1)}\left[1-\left(\frac{a}{b}\right)^{(2l+1)}\right]^{-1}a^{-(l+1)}\right] \\
&= \frac{2l+1}{2}\frac{VI_l}{a^l}\left[1-\left[1-\left(-\frac{a}{b}\right)^l\right]\left[1-\left(\frac{a}{b}\right)^{(2l+1)}\right]^{-1}\right] \\
&= \frac{2l+1}{2}\frac{VI_l}{b^{(2l+1)}-a^{(2l+1)}}\left[(-1)^l b^{(l+1)} - a^{(l+1)}\right]
\end{aligned}
$$

c) Which coefficients in the series vanish for $r < a$ and which vanish for $r > b$? Why?
All $A_l \equiv 0$ for $r > b$ and all $B_l \equiv 0$ for $r < a$, since otherwise the potential diverges.

4. Consider a cylinder of radius R and length L that is uniformly charged with charge density ρ. The cylinder rotates with a uniform angular velocity ω around the z-axis, which is also the center axis of the cylinder, as shown in the figure below



a) Compute the current density, J, as a function of distance, r, from the center of the cylinder.
b) Compute the magnetic induction, $\vec{B}$, along the z-axis at $\vec{r} = (0, 0, h)$.
c) If the charge on the cylinder is kept the same, but redistributed such that the charge density obeys $\rho(r) = \alpha r^n$, do you expect that the
resulting magnetic field is smaller or larger than that you computed in part b)? Explain!

Solution:
a) Compute the current density, $\vec{J}$, as a function of distance, r, from the center of the cylinder.
The current density, J, is given by

$$
\begin{aligned}
\vec{J}(r) &= \rho\vec{v}(r) = \rho\vec{\omega}\times\vec{r} \\
J(\vec{r}') &= \left|\vec{J}(r)\right| = \rho\omega r
\end{aligned}
$$

b) Compute the magnetic induction, $\vec{B}$, along the z-axis at $\vec{r} = (0, 0, h)$.
The magnetic induction is given using $\vec{r} = (0, 0, h)$ by

$$
\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\vec{J}(\vec{r}')\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}
$$

and hence

$$B_z(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\, J(\vec{r}')\frac{\cos\alpha}{|\vec{r}-\vec{r}'|^2} = \frac{\mu_0}{4\pi}\int_0^L dz'\int_0^{2\pi}d\varphi'\int_0^R r'dr'\,\rho\,\omega\, r'\frac{r'}{\left[(h+z')^2+(r')^2\right]^{3/2}}$$

$$= \frac{\mu_0}{2}\rho\,\omega\int_0^L dz'\int_0^R dr'\frac{(r')^3}{\left[(h+z')^2+(r')^2\right]^{3/2}} = \frac{\mu_0}{2}\rho\,\omega\int_0^L dz'\left[\frac{2(h+z')^2+R^2}{\left[(h+z')^2+R^2\right]^{1/2}}-2(h+z')\right]$$

$$= \frac{\mu_0}{2}\rho\,\omega\left[(h+L)\left(\sqrt{(h+L)^2+R^2}-(h+L)\right)-h\left(\sqrt{h^2+R^2}-h\right)\right]$$

c) If the charge on the cylinder is kept the same, but redistributed such that the charge density obeys $\rho(r) = \alpha r^n$, do you expect that the resulting magnetic field is smaller or larger than that you computed in part b)? Explain!

Since $\rho(r) = \alpha r^n$, more charge is located close to the perimeter of the cylinder, leading to an increase in $J(r)$ (more specifically, the averaged $\overline{J(r)}$ increases, since

$$\overline{J(r)} = \frac{1}{R}\int_0^R rdrJ(r) = \frac{1}{R}\int_0^R rdr\rho\,\omega\, r = \frac{\alpha\omega}{R}\int_0^R dr\, r^{n+2} = \frac{\alpha\omega}{R}\frac{R^{n+3}}{n+3} = \frac{Q\omega}{2\pi}\frac{n+2}{n+3}$$

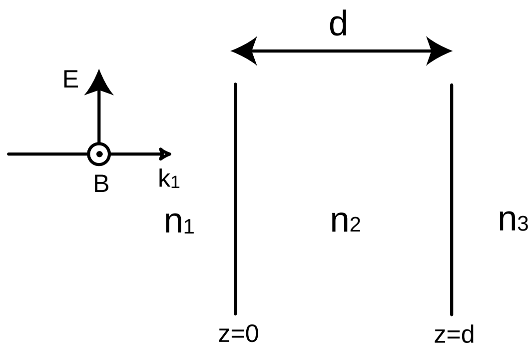
where I used

$$2\pi\int_0^R rdr\rho(r) = Q = 2\pi\alpha\int_0^R dr\, r^{n+1} = 2\pi\alpha\frac{R^{n+2}}{n+2}$$

$$\Rightarrow \alpha = \frac{(n+2)Q}{2\pi R^{n+2}}$$

On the other hand, the charge is moved further away from $\vec{r} = (0,0,h)$. This two effects oppose each other, so in general it depends on h and n whether the magnetic induction is increased or decreased. However, for $h >> R$, the magnetic induction increases with increasing n.

5. An electromagnetic plane wave is incident perpendicular to a layered interface, as shown in the figure below. The indices of refraction of the three media is n_1, n_2 and n_3 while the permeability of all three regions is the same, μ_0. The thickness of the intermediate layer is d. Each of the other media is semi-infinite.



a) State the boundary conditions at both interfaces in terms of the electric fields.

b) Compute the ratio between the incident electric field in medium 1 and the transmitted electric field in medium 3, i..e, compute $|E_i/E_t|^2$.

c) If the thickness d is varied, the ratio $|E_i/E_t|^2$ oscillates. What is the period of the oscillation? Assuming $n_1 < n_2 < n_3$, for which values of d is $|E_i/E_t|^2$ the smallest?

Solution:

a) State the boundary conditions at both interfaces in terms of the electric fields.

The EM wave contains only components that are perpendicular to the interface. In region 1, there is an incoming and a reflected wave, in region 2 there is a right-moving and a left-moving wave, and in region 3, there is only a transmitted wave. Thus the boundary conditions at $z = 0$ are

$$E_i + E_r = E_+ + E_-$$
$$\frac{E_i - E_r}{c_1} = \frac{E_+ - E_-}{c_2} \Rightarrow E_i - E_r = \frac{n_2}{n_1}(E_+ - E_-)$$

and at $z = d$ we have

$$E_+ e^{ik_2 d} + E_- e^{-ik_2 d} = E_t e^{ik_3 d}$$
$$\frac{E_+ e^{ik_2 d} - E_- e^{-ik_2 d}}{c_2} = \frac{E_t e^{ik_3 d}}{c_3} \Rightarrow E_+ e^{ik_2 d} - E_- e^{-ik_2 d} = \frac{n_3}{n_2} E_t e^{ik_3 d}$$

b) Compute the ratio between the incident electric field in medium 1 and the transmitted electric field in medium 3, i..e, compute $|E_i/E_t|^2$.

From the last two equations, I obtain

$$E_+ = \frac{1}{2}\left(1 + \frac{n_3}{n_2}\right) E_t e^{ik_3 d} e^{-ik_2 d}$$
$$E_- = \frac{1}{2}\left(1 - \frac{n_3}{n_2}\right) E_t e^{ik_3 d} e^{ik_2 d}$$

and from the first two equations

$$\begin{aligned}
2E_i &= \left(1 + \frac{n_2}{n_1}\right) E_+ + \left(1 - \frac{n_2}{n_1}\right) E_- \\
&= \left(1 + \frac{n_2}{n_1}\right) \frac{1}{2}\left(1 + \frac{n_3}{n_2}\right) E_t e^{ik_3 d} e^{-ik_2 d} + \left(1 - \frac{n_2}{n_1}\right) \frac{1}{2}\left(1 - \frac{n_3}{n_2}\right) E_t e^{ik_3 d} e^{ik_2 d}
\end{aligned}$$

and thus

$$4\frac{E_i}{E_t} = e^{ik_3 d} e^{-ik_2 d}\left[\left(1 + \frac{n_2}{n_1}\right)\left(1 + \frac{n_3}{n_2}\right) + \left(1 - \frac{n_2}{n_1}\right)\left(1 - \frac{n_3}{n_2}\right) e^{i2k_2 d}\right]$$

and hence

$$\begin{aligned}
16\left|\frac{E_i}{E_t}\right|^2 &= \left[\left(1 + \frac{n_2}{n_1}\right)\left(1 + \frac{n_3}{n_2}\right) + \left(1 - \frac{n_2}{n_1}\right)\left(1 - \frac{n_3}{n_2}\right) e^{i2k_2 d}\right] \\
&\quad \times \left[\left(1 + \frac{n_2}{n_1}\right)\left(1 + \frac{n_3}{n_2}\right) + \left(1 - \frac{n_2}{n_1}\right)\left(1 - \frac{n_3}{n_2}\right) e^{-i2k_2 d}\right] \\
&= \left(1 + \frac{n_2}{n_1}\right)^2\left(1 + \frac{n_3}{n_2}\right)^2 + \left(1 - \frac{n_2}{n_1}\right)^2\left(1 - \frac{n_3}{n_2}\right)^2 \\
&\quad + \left(1 + \frac{n_2}{n_1}\right)\left(1 + \frac{n_3}{n_2}\right)\left(1 - \frac{n_2}{n_1}\right)\left(1 - \frac{n_3}{n_2}\right) 2\cos(2k_2 d) \\
&= \left(1 + \frac{n_2}{n_1}\right)^2\left(1 + \frac{n_3}{n_2}\right)^2 + \left(1 - \frac{n_2}{n_1}\right)^2\left(1 - \frac{n_3}{n_2}\right)^2 \\
&\quad + 2\left[1 - \left(\frac{n_2}{n_1}\right)^2\right]\left[1 - \left(\frac{n_3}{n_2}\right)^2\right][1 - 2\sin^2(k_2 d)] \\
&= 4\left(1 + \frac{n_2}{n_1}\frac{n_3}{n_2}\right)^2 - 4\left[1 - \left(\frac{n_2}{n_1}\right)^2\right]\left[1 - \left(\frac{n_3}{n_2}\right)^2\right]\sin^2(k_2 d)
\end{aligned}$$

or alternatively

$$\left|\frac{E_i}{E_t}\right|^2 = \frac{1}{4}\left[\left(1+\frac{n_2}{n_1}\frac{n_3}{n_2}\right)^2 - \left[1-\left(\frac{n_2}{n_1}\right)^2\right]\left[1-\left(\frac{n_3}{n_2}\right)^2\right]\sin^2(k_2 d)\right]$$

c) If the thickness d is varied, the ratio $|E_i/E_t|^2$ oscillates. What is the period of the oscillation? Assuming $n_1 < n_2 < n_3$, for which values of d is $|E_i/E_t|^2$ the smallest?

The period of the oscillation is λ_2, and $|E_i/E_t|^2$ is the smallest for $d = (2n+1)\lambda_2/2$.

Mathematical Formulae

Definitions

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\vec{E}(\vec{r}) = -\nabla\Phi(\vec{r})$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \vec{J}(\vec{r}') \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\Delta\Phi = -\int \vec{E}(\vec{r}) \cdot d\vec{r}$$

$$C = \frac{Q}{\Delta\Phi}; \qquad \sigma = -\varepsilon_0 \frac{\partial\Phi}{\partial n}$$

$$\nabla\vec{E} = \frac{\rho}{\varepsilon_0}; \qquad \nabla\vec{B} = 0$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}; \qquad \nabla\times\vec{B} = \mu_0\vec{J}$$

Integrals and Series

$$\int_0^{2\pi} \frac{d\varphi}{\sqrt{a-b\cos\varphi}} = \frac{1}{a-b}K\left[\frac{-2b}{a-b}\right]$$ where K is the complete elliptic integral

$$\int_0^b \frac{x^3}{[a^2+x^2]^{3/2}}dx = \frac{2a^2+b^2}{[a^2+b^2]^{1/2}} - 2a$$

$$\int_0^c dx\left[\frac{2(a+x)^2+b^2}{\left[(a+x)^2+b^2\right]^{1/2}}-2(a+x)\right]=\left[(a+c)\left(\sqrt{(a+c)^2+b^2}-(a+c)\right)-a\left(\sqrt{a^2+b^2}-a\right)\right]$$

$$\int_0^1 dx\,P_l(x)=\begin{cases} 0 & \text{for even } l \\ 1 & \text{for } l=0 \\ (-1)^{\frac{l-1}{2}}\frac{(l+1)(l-1)!}{2^{l+1}\left[\left(\frac{l+1}{2}\right)!\right]^2} & \text{for odd } l \end{cases}$$

$$\int_{-1}^0 dx\,P_l(x)=(-1)^l\int_0^1 dx\,P_l(x)$$

$$\int_{-1}^1 dx\,[P_l(x)]^2=\frac{2}{2l+1}$$

$$\Phi(r,\theta)=\sum_n\left[A_n r^n+B_n r^{-(n+1)}\right]P_n(\cos\theta)$$