University of Illinois at Chicago
Department of Physics

Electricity & Magnetism
Qualifying Examination

January 7, 2008
9.00 am – 12:00 pm

Full credit can be achieved from completely correct answers to <u>4 questions</u>. If the student attempts all 5 questions, all of the answers will be graded, and the <u>top 4 scores</u> will be counted toward the exam's total score.

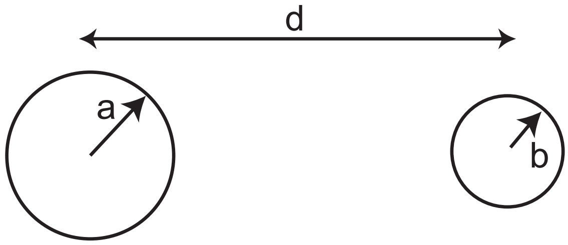
1. Consider a ring of uniform charge density λ and radius R that lies within the xy-plane. The origin of the coordinate systems is located at the center of the ring.

a) Give the potential at the point $\vec{P} = (\rho_0, \varphi, z)$ in terms of $\lambda, R, \rho_0, \varphi$, and z.

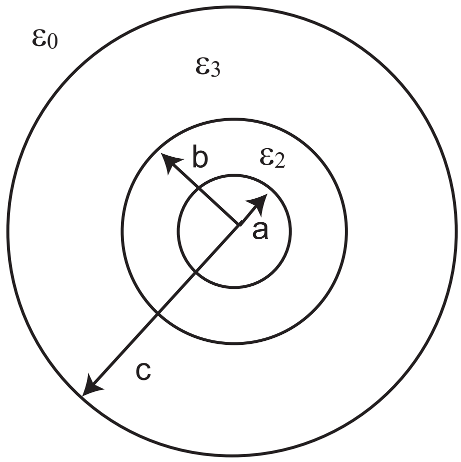
b) We next put a conducting plane into the $z = d$ plane. The potential of the conducting plane is fixed at $V = 0$. Compute the total potential at a point $\vec{P} = (\rho_0, \varphi, z)$.

c) Give an explicit form of the induced charge density at $\vec{P} = (0, 0, d)$? Your final answer should contain no integrals or derivatives.

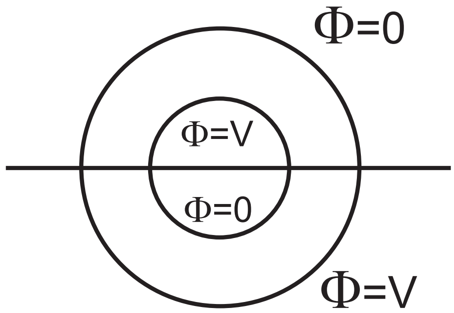
2. a) Using Gauss' law, compute the capacitance per unit length of two infinitely long cylindrical conductors of radii a and b that are parallel and separated by a distance $d \gg a, b$, as shown in the figure below



b) Consider next two infinitely long concentric cylinders, as shown in the figure below. The inner cylinder of radius a is a conductor with linear charge density $\lambda_1 > 0$. The second cylinder with inner radius b and outer radius c consists of a material with permitivity ε_3 and is uniformly charged with line charge density $\lambda_3 < 0$ ($\lambda_1 > |\lambda_3|$). The space between the two cylinders (i.e., $a < r < b$) is filled with a medium of permitivity ε_2. The medium outside the outer cylinder possesses the permitivity ε_0. Compute the potential difference between a point at $|\vec{r}| = 2c$ and the center of the inner cylinder.



3. Two concentric spheres have radii a and b ($b > a$) and each is divided into two hemispheres by the same horizontal plane, as shown below. The potential of the upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere is kept at V. The other hemispheres are at zero potential.

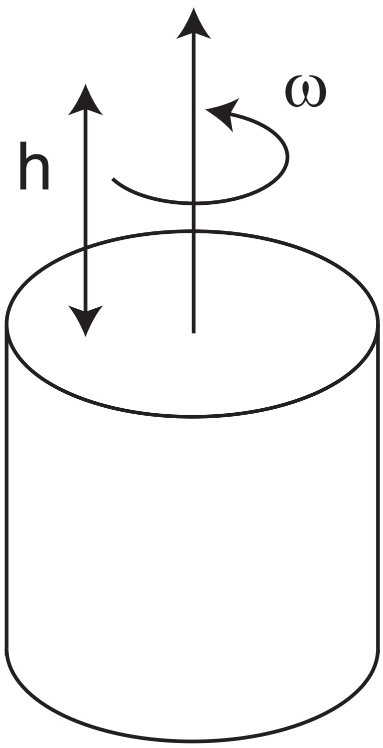


a) Derive an explicit form of the boundary conditions, using a series expansion of the potential in Legendre Polynomials.

b) Derive an expression for the coefficients (to all order) in the series expansion of the potential in the region $a < r < b$. Give the explicit form of the coefficients up to $l = 2$.

c) Which coefficients in the series vanish for $r < a$ and which vanish for $r > b$? Why?

4. Consider a cylinder of radius R and length L that is uniformly charged with charge density ρ. The cylinder rotates with a uniform angular velocity ω around the z-axis, which is also the center axis of the cylinder, as shown in the figure below

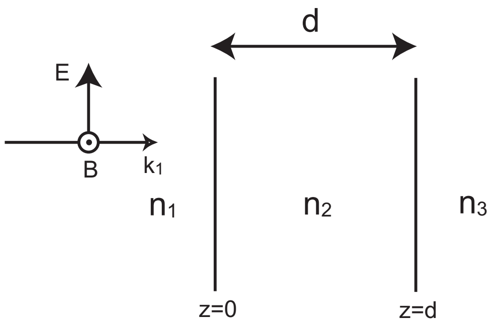


a) Compute the current density, J, as a function of distance, r, from the center of the cylinder.

b) Compute the magnetic induction, $\vec{B}$, along the z-axis at $\vec{r} = (0, 0, h)$.

c) If the charge on the cylinder is kept the same, but redistributed such that the charge density obeys $\rho(r) = \alpha r^n$, do you expect that the resulting magnetic field is smaller or larger than that you computed in part b)? Explain!

5. An electromagnetic plane wave is incident perpendicular to a layered interface, as shown in the figure below. The indices of refraction of the three media is n_1, n_2 and n_3 while the permeability of all three regions is the same, μ_0. The thickness of the intermediate layer is d. Each of the other media is semi-infinite.



a) State the boundary conditions at both interfaces in terms of the electric fields.

b) Compute the ratio between the incident electric field in medium 1 and the transmitted electric field in medium 3, i..e, compute $|E_i/E_t|^2$.

c) If the thickness d is varied, the ratio $|E_i/E_t|^2$ oscillates. What is the period of the oscillation? Assuming $n_1 < n_2 < n_3$, for which values of d is $|E_i/E_t|^2$ the smallest?

I. MATHEMATICAL FORMULAE

A. Definitions

$$
\begin{aligned}
\Phi(\vec{r}) &= \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \\
\vec{E}\,(\vec{r}) &= -\nabla\Phi(\vec{r}) \\
\vec{B}\,(\vec{r}) &= \frac{\mu_0}{4\pi}\int d^3r'\,\vec{J}(\vec{r}')\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \\
\vec{A}\,(\vec{r}) &= \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}
\end{aligned}
$$

$$\Delta\Phi = -\int \vec{E}\,(\vec{r})\cdot d\vec{r}$$

$$C = \frac{Q}{\Delta\Phi};\qquad \sigma = -\varepsilon_0\frac{\partial\Phi}{\partial n}$$

$$
\begin{aligned}
\nabla\vec{E} &= \frac{\rho}{\varepsilon_0};\qquad \nabla\vec{B} = 0 \\
\nabla\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t};\qquad \nabla\times\vec{B} = \mu_0\vec{J}
\end{aligned}
$$

B. Integrals and Series

$$\int_0^{2\pi}\frac{d\varphi}{\sqrt{a-b\cos\varphi}} = \frac{1}{a-b}K\left[\frac{-2b}{a-b}\right]\qquad \text{where } K \text{ is the complete elliptic integral}$$

$$\int_0^b \frac{x^3}{\left[a^2+x^2\right]^{3/2}}dx = \frac{2a^2+b^2}{\left[a^2+b^2\right]^{1/2}} - 2a$$

$$\int_0^c dx\left[\frac{2\,(a+x)^2+b^2}{\left[(a+x)^2+b^2\right]^{1/2}} - 2\,(a+x)\right] = (a+c)\left[\sqrt{(a+c)^2+b^2}-(a+c)\right] - a\left[\sqrt{a^2+b^2}-a\right]$$

$$
\begin{aligned}
\int_0^1 dx\;P_l(x) &= \begin{cases} 0 & \text{for even } l \\ 1 & \text{for } l=0 \\ (-1)^{\frac{l-1}{2}}\,\frac{(l+1)(l-1)!}{2^{l+1}\left[\left(\frac{l+1}{2}\right)!\right]^2} & \text{for odd } l \end{cases} \\
\int_{-1}^0 dx\,P_l(x) &= (-1)^l\int_0^1 dx\;P_l(x)
\end{aligned}
$$

$$\int_{-1}^1 dx\;\left[P_l(x)\right]^2 = \frac{2}{2l+1}$$

$$\Phi(r,\theta) = \sum_n\left[A_n r^n + B_n r^{-(n+1)}\right]P_n\left(\cos\theta\right)$$