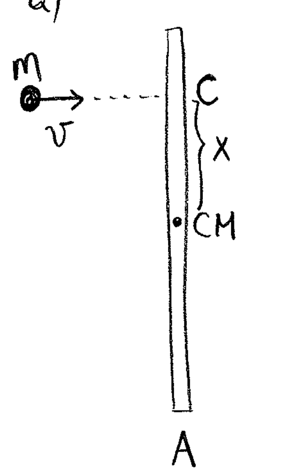
Q1) a)




Let $v'$ be the speed of m after the collision, and let $\omega$ be the angular speed of the rod after the collision, and let $V_{CM}$ be the linear speed of the CM of the rod after the collision $\Rightarrow$

Linear momentum conservation $\Rightarrow$ $mv = mv' + MV_{CM}$ (I)

Angular momentum conservation (wrt CM) $\Rightarrow$ $mvx = mv'x + \frac{1}{12}ML^2\omega$ (II)

If A is to be the point of pure rotation after the collision $\Rightarrow$ $V_A = 0$

$$\Rightarrow V_{CM} + \left(-\omega\frac{L}{2}\right) = 0 \Rightarrow V_{CM} = \frac{\omega L}{2} \quad \text{(III)}$$

From (I) & (II) & (III) $\Rightarrow$ $\left.\begin{array}{l} m(v-v') = MV_{CM} \\ m(v-v')x = \frac{1}{12}ML^2\frac{2V_{CM}}{L} \end{array}\right\}$ take ratios $\Rightarrow$ $\underline{\underline{X = \frac{L}{6}}}$ $\Rightarrow$ $\underline{\underline{AC = \frac{2L}{3}}}$

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b) $mv = mv' + MV_{CM}$ $\qquad \vec{p}$-conservation

$mv.\frac{L}{4} = mv'\frac{L}{4} + \frac{1}{12}ML^2\omega$ $\qquad \vec{L}$-conservation (wrt CM)

$\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + \frac{1}{2}MV_{CM}^2 + \frac{1}{2}\left(\frac{1}{12}ML^2\right)\omega^2$ (Energy conservation, elastic)

With $m = M$ we have

$$v = v' + V_{CM} \quad \text{(I)}$$

$$3v = 3v' + L\omega \quad \text{(II)}$$

$$12v^2 = 12v'^2 + 12V_{CM}^2 + \omega^2L^2 \quad \text{(III)}$$

(I) & (II) $\Rightarrow$ $V_{CM} = \frac{\omega L}{3}$ ; (III) $\Rightarrow$ $12\underbrace{(v-v')}_{\frac{\omega L}{3}}(v+v') = 12\left(\frac{\omega L}{3}\right)^2 + \omega^2L^2 \Rightarrow$

$$4\omega L(v+v') = \frac{21}{9}\omega^2 L^2 \Rightarrow v+v' = \frac{7}{12}\omega L \quad \text{(solving this with II gives)}$$

$$\underline{\underline{v' = \frac{3}{11}v}} \; ; \; \underline{\underline{\omega L = \frac{24}{11}v}} \; ; \; \underline{\underline{v_{CM} = \frac{8}{11}v}}$$

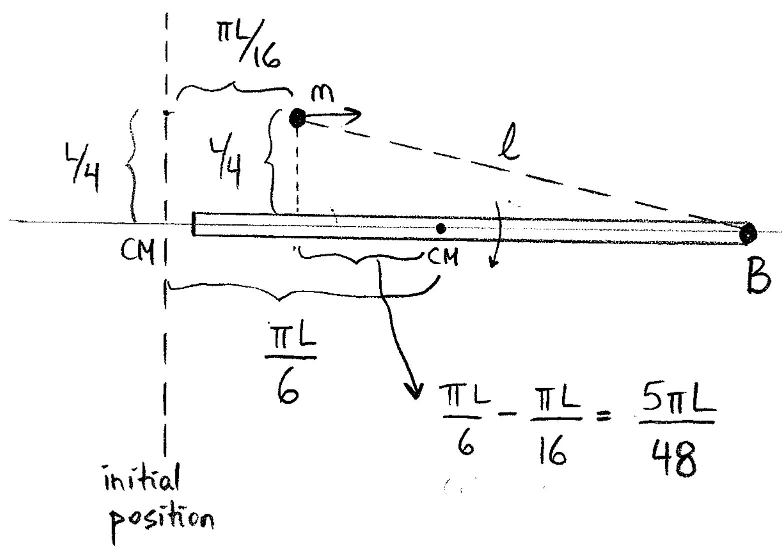
(notice $v_{CM} > v'$ so that the rod leads)

After the collision, the rod rotates with angular speed $\omega$ (while translating with CM speed $v_{CM}$). It becomes aligned along $\hat{x}$ (1st time) at time $t$ when $\omega t = \frac{\pi}{2}$, i.e when $t = \frac{\pi}{2\omega} = \frac{11\pi L}{48v}$

During this time, the object moves by $\Delta x' = v't = \frac{3}{11}v \cdot \frac{11\pi L}{48v} = \frac{\pi L}{16}$

the CM moves by $\Delta x_{CM} = \frac{8}{11}v \cdot \frac{11\pi L}{48v} = \frac{\pi L}{6} > \frac{L}{2}$ (point A has cleared initial position)

so, the configuration is something like




$$\frac{\pi L}{6} - \frac{\pi L}{16} = \frac{5\pi L}{48}$$

$$\Rightarrow l = \sqrt{\left(\frac{L}{4}\right)^2 + \left(\frac{5\pi L}{48} + \frac{L}{2}\right)^2}$$

$$\approx 0.864L$$

Q2)

(a) The restoring force $kx$ (along $-\hat{r}$) should provide the centripetal acceleration, i.e $k\left(\frac{5r_0}{4} - r_0\right) = \frac{mv_0^2}{\left(\frac{5r_0}{4}\right)} \Rightarrow v_0 = \frac{r_0}{4}\sqrt{\frac{5k}{m}}$

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(b) $T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right)$, $V = \frac{1}{2}k(r-r_0)^2$

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{1}{2}k(r-r_0)^2$$

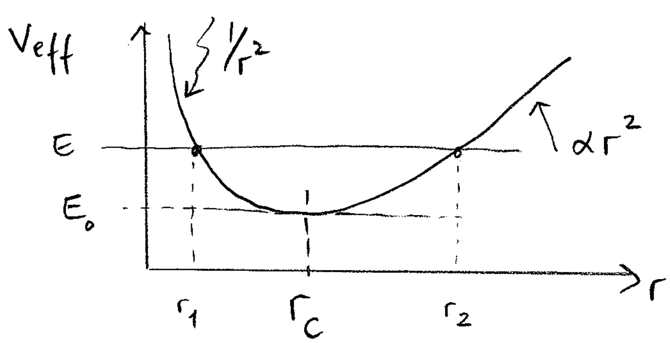
(i) $\frac{\partial \mathcal{L}}{\partial \phi} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$ is conserved i.e $mr^2\dot{\phi} = \text{constant}$, which is clearly the angular momentum (along z)

call $L_z = mr^2\dot{\phi} \Rightarrow \dot{\phi} = \frac{L_z}{mr^2}$

(ii) $E_{total} = T + V = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\left(\frac{L_z}{mr^2}\right)^2 + \frac{1}{2}k(r-r_0)^2$

$$\Rightarrow V_{eff}(r) = \frac{L_z^2}{2mr^2} + \frac{1}{2}k(r-r_0)^2$$

(iii) As $r \to 0$ $V_{eff} \sim 1/r^2$, As $r \to \infty$ $V_{eff}(r) \sim r^2 \Rightarrow$ should have a minimum




At a given E, in principle there are 2 turning points. but if $E = E_0 \Rightarrow$ single $r = r_c$ and $V_{eff}$ exhibits a minimum $\Rightarrow$ stable circular orbit

$\Rightarrow$ Stable circular orbit at $r = r_c$ where $\left.\frac{\partial V_{eff}}{\partial r}\right)_{r=r_c} = 0$

(iv) $\left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_c}=0 \Rightarrow -\frac{L_z^2}{mr_c^3}+k(r_c-r_o)=0 \Rightarrow$ If a circular orbit at $r_c=\frac{5r_o}{4}$ is required $\Rightarrow$

$$L_z^2=m\left(\frac{5r_o}{4}\right)^3 k\left(\frac{r_o}{4}\right) \Rightarrow L_z=\frac{5r_o^2}{16}\sqrt{5mk}$$

but $L_z=mrv \Rightarrow v=\frac{L_z}{mr}=\frac{5r_o^2\sqrt{5mk}}{16}\frac{1}{m\frac{5r_o}{4}}=\frac{r_o}{4}\sqrt{\frac{5k}{m}}$ ✓

(same as in (a))

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(c) For radial coordinate, Lagrange's equation of yields

$$m\ddot{r}-\frac{L_z^2}{mr^3}+k(r-r_o)=0 \Rightarrow \text{Let } r=r_o+x \ (x \ll r_o), \text{ we have}$$

$$m\ddot{x}-\frac{L_z^2}{mr_o^3}\left(1+\frac{x}{r_o}\right)^{-3}+kx=0 \Rightarrow m\ddot{x}-\frac{L_z^2}{mr_o^3}\left(1-\frac{3x}{r_o}+\cdots\right)+kx=0$$

$$\Rightarrow m\ddot{x}+\left(k+\frac{3L_z^2}{mr_o^4}\right)x-\frac{L_z^2}{mr_o^3}=0$$

OR Expand $V_{eff}(r)$ around $r=r_c$

$$V_{eff}(r)=V_{eff}(r_c)+\frac{1}{2}\left.\frac{d^2V}{dr^2}\right|_{r=r_c}(r-r_c)^2$$

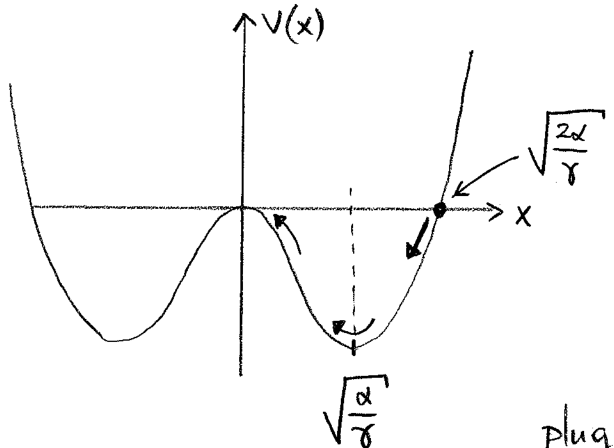
and set $\frac{1}{2}V''(r_c)=\frac{1}{2}m\omega^2$ to find $\omega$

$$\Rightarrow \omega=\left(\frac{k+\frac{3L_z^2}{mr_o^4}}{m}\right)^{1/2}=\sqrt{\frac{k+3\frac{(25r_o^4)(5mk)}{256mr_o^4}}{m}}\approx 1.57\sqrt{\frac{k}{m}}$$

Q3)

a) Conservative force field ⇒ $F = \frac{-\partial V}{\partial x}$ , i.e $\ddot{x} + \frac{\partial V}{\partial x} = 0$ $(m=1)$

⇒ $V(x) = \frac{-\alpha}{2}x^2 + \frac{\gamma}{4}x^4 + C$ constant can be set $= 0$ (just a constant shift)




$$E = \frac{1}{2}\dot{x}^2 + V(x)$$

$$\dot{x}(t) = -\sqrt{\frac{2\alpha}{\gamma}}\,\frac{\sinh(\sqrt{\alpha}\,t)}{\cosh^2(\sqrt{\alpha}\,t)}\sqrt{\alpha}$$

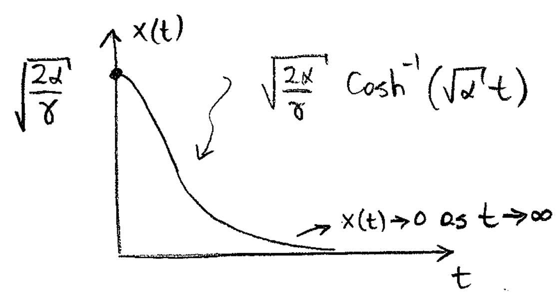
plug in ⇒

$$E = \frac{1}{2}\,\frac{2\alpha^2}{\gamma}\,\frac{\sinh^2(\sqrt{\alpha}\,t)}{\cosh^4(\sqrt{\alpha}\,t)} - \frac{\alpha}{2}\,\frac{2\alpha}{\gamma}\,\frac{1}{\cosh^2(\sqrt{\alpha}\,t)} + \frac{\gamma}{4}\,\frac{4\alpha^2}{\gamma}\,\frac{1}{\cosh^4(\sqrt{\alpha}\,t)}$$

Using $\cosh^2 x - \sinh^2 x = 1$

$$E = \frac{\alpha^2}{\gamma}\left[\frac{\cosh^2(\sqrt{\alpha}\,t) - 1}{\cosh^4(\sqrt{\alpha}\,t)} - \frac{1}{\cosh^2\sqrt{\alpha}\,t} + \frac{1}{\cosh^4(\sqrt{\alpha}\,t}\right] = 0$$

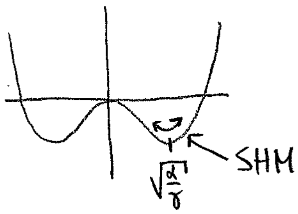
So this is the $E=0$ solution. A particle starting with $\dot{x}(0) = 0$ from $x(0) = \sqrt{\frac{2\alpha}{\gamma}}$ (see above) will roll down the well and asymptotically reach $x=0$ as $t \to \infty$, will never reach $x<0$ consistent with $[\cosh(\sqrt{\alpha}\,t)]^{-1}$ solution.



(b) Notice that $V'(x)=0 \Rightarrow x = \mp\sqrt{\frac{\alpha}{\gamma}}$ ∴ Near the minima ⇒ expect simple Harmonic motion

$$V''\left(x=\sqrt{\tfrac{\alpha}{\gamma}}\right) = -2\alpha + 3\gamma x^2 \Big|_{x=\sqrt{\frac{\alpha}{\gamma}}} = \alpha \text{ , as expected}$$

$$\Rightarrow \omega^2 = \alpha \Rightarrow \omega = \sqrt{\alpha} \Rightarrow x(t) = \overset{\text{amplitude}=\sqrt{\frac{\alpha}{\gamma}}\,\delta}{A}\cos\omega t \;\; (\text{since } \dot{x}(0)=0) \Rightarrow \underline{\underline{x(t) = \sqrt{\frac{\alpha}{\gamma}}\,\delta\cos\sqrt{\alpha}\,t}}$$

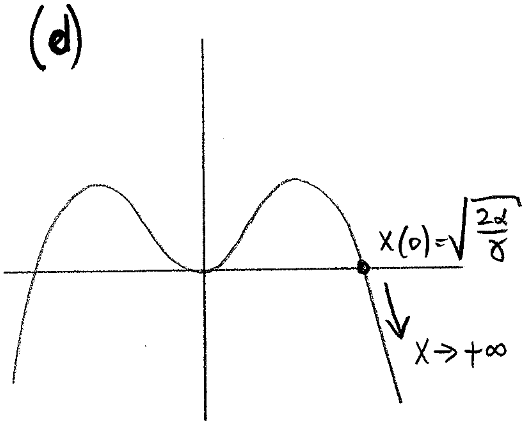



(c) Solution for $\ddot{x} - \alpha x + \gamma x^3 = 0$ is $x(t) = \dfrac{\sqrt{\frac{2\alpha}{\gamma}}}{\cosh(\sqrt{\alpha}\,t)}$

let $\alpha \to -\alpha$, $\gamma \to -\gamma$ we have $\ddot{x} + \alpha x - \gamma x^3 = 0 \Rightarrow x(t) = \dfrac{\sqrt{\frac{2\alpha}{\gamma}}}{\cosh(i\sqrt{\alpha}\,t)}$

$$\Rightarrow \underline{\underline{x(t) = \frac{\sqrt{\frac{2\alpha}{\gamma}}}{\cos(\sqrt{\alpha}\,t)}}}$$

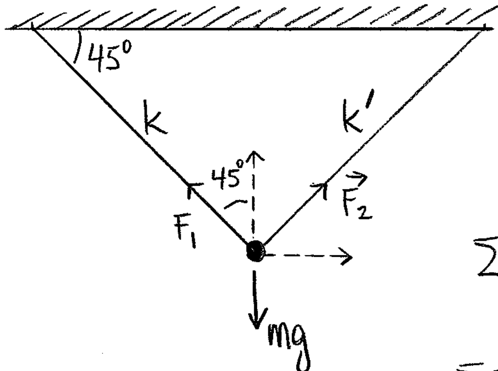
(d)




$$x(t) = \frac{\sqrt{\frac{2\alpha}{\gamma}}}{\cos(\sqrt{\alpha}\,t)} \to \infty \text{ when } \sqrt{\alpha}\,t = \pi/2$$

$$\Rightarrow \underline{\underline{t = \frac{\pi}{2\sqrt{\alpha}}}}$$

Q.4 a)




$$|\vec{F}_1| = k\left(L - \frac{L}{2}\right) = k\frac{L}{2} \; ; \; |\vec{F}_2| = k'\left(L - \frac{L}{3}\right) = \frac{2k'L}{3}$$

$$\sum F_x = 0 \Rightarrow k\frac{L}{2}\frac{1}{\sqrt{2}} = \frac{2k'L}{3}\frac{1}{\sqrt{2}} \Rightarrow k' = \frac{3k}{4}$$

$$\sum F_y = 0 \Rightarrow mg = \frac{kL}{2}\cdot\frac{1}{\sqrt{2}} + \frac{2}{3}\,\frac{3k}{4}L\cdot\frac{1}{\sqrt{2}}$$

$$\Rightarrow mg = \frac{kL}{\sqrt{2}}$$

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b) When $m$ is moved to $\vec{r}$, extended length of spring 1 $= \overline{PA} - \frac{L}{2}$ ← natural length

$$\Rightarrow U_A = \frac{1}{2}k\left(L + \frac{r^2}{2L} - \frac{\vec{R}_A\cdot\vec{r}}{L} - \frac{(\vec{R}_A\cdot\vec{r})^2}{2L^3} - \frac{L}{2}\right)^2$$

$$= \frac{1}{2}k\left(\frac{L}{2} + \frac{r^2}{2L} - \frac{\vec{R}_A\cdot\vec{r}}{L} - \frac{(\vec{R}_A\cdot\vec{r})^2}{2L^3}\right)^2$$

. Retaining only terms up to $r^2$ brings 2 direct squares, and 3 cross terms

$$= \frac{1}{2}k\Bigg(\underbrace{\frac{L^2}{4} + \frac{(\vec{R}_A\cdot\vec{r})^2}{L^2}}_{\text{direct terms}} + \underbrace{\frac{r^2}{2} - \vec{r}.\vec{R}_A - \frac{(\vec{R}_A\cdot\vec{r})^2}{2L^2}}_{\text{Cross terms of } \frac{L}{2}}\Bigg) = \underline{\underline{\frac{1}{2}k\left(\frac{L^2}{4} + \frac{r^2}{2} - \vec{R}_A\cdot\vec{r} + \frac{(\vec{R}_A\cdot\vec{r})^2}{2L^2}\right)}}$$

Similarly, extended length of spring 2 $= \overline{PB} - \frac{L}{3}$

$$\Rightarrow U_B = \frac{1}{2}k'\left(L + \frac{r^2}{2L} - \frac{\vec{R}_B\cdot\vec{r}}{L} - \frac{(\vec{R}_B\cdot\vec{r})^2}{2L^3} - \frac{L}{3}\right)^2 = \frac{3k}{8}\left(\frac{2L}{3} + \frac{r^2}{2L} - \frac{\vec{R}_B\cdot\vec{r}}{L} - \frac{(\vec{R}_B\cdot\vec{r})^2}{2L^3}\right)^2$$

$$= \frac{3k}{8}\left(\frac{4L^2}{9} + \frac{(\vec{R}_B\cdot\vec{r})^2}{L^2} + \frac{2r^2}{3} - \frac{4(\vec{R}_B\cdot\vec{r})}{3} - \frac{2(\vec{R}_B\cdot\vec{r})^2}{3L^2}\right) = \frac{3k}{8}\left(\frac{4L^2}{9} + \frac{2r^2}{3} - \frac{4(\vec{R}_B\cdot\vec{r})}{3} + \frac{(\vec{R}_B\cdot\vec{r})^2}{3L^2}\right)$$

$$= \underline{\underline{\frac{k}{2}\left(\frac{L^2}{3} + \frac{r^2}{2} - (\vec{R}_B\cdot\vec{r}) + \frac{(\vec{R}_B\cdot\vec{r})^2}{4L^2}\right)}}$$

(c) $\vec{R}_A = \frac{L}{\sqrt{2}}(-1,1) \Rightarrow \vec{r}\cdot\vec{R}_A = \frac{L}{\sqrt{2}}(-x+y)$ $\downarrow$ $\vec{R}_B = \frac{L}{\sqrt{2}}(1,1) \Rightarrow \vec{r}\cdot\vec{R}_B = \frac{L}{\sqrt{2}}(x+y)$

$U = U_A + U_B + mgy$

$$= \frac{k}{2}\left(\frac{L^2}{4} + \frac{L^2}{3} + \frac{x^2+y^2}{2}\otimes 2 - \frac{L}{\sqrt{2}}(\cancel{-x}+y) - \frac{L}{\sqrt{2}}(\cancel{x}+y) + \frac{1}{4}(-x+y)^2 + \frac{1}{8}(x+y)^2\right)$$

$$= \frac{k}{2}\left(\frac{7L^2}{12} + x^2+y^2 - \cancel{yL\sqrt{2}} + \frac{1}{4}\left(\frac{3x^2}{2}+\frac{3y^2}{2} - xy\right)\right) + \cancel{mgy} \quad \left(\text{use } mg = \frac{kL}{\sqrt{2}}\right)$$

$$= \frac{k}{2}\left(\frac{7L^2}{12} + \frac{11}{8}(x^2+y^2) - \frac{xy}{4}\right)$$

$$\Rightarrow L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{k}{2}\left(\frac{7L^2}{12} + \frac{11}{8}(x^2+y^2) - \frac{xy}{4}\right)$$

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(d)
$$\left.\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} + \frac{11k}{8}x - \frac{ky}{8} = 0 \\ &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \Rightarrow m\ddot{y} + \frac{11k}{8}y - \frac{kx}{8} = 0 \end{aligned}\right\} \begin{aligned} x &= A_1 \cos\omega t \\ y &= A_2 \cos\omega t \end{aligned} \text{ plug in}$$

$$\begin{pmatrix} -m\omega^2 + \frac{11k}{8} & -\frac{k}{8} \\ -\frac{k}{8} & -m\omega^2 + \frac{11k}{8} \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \vec{0} \Rightarrow \det\begin{vmatrix} -m\omega^2 + \frac{11k}{8} & -\frac{k}{8} \\ -\frac{k}{8} & -m\omega^2 + \frac{11k}{8} \end{vmatrix} = 0 \quad \text{set } \lambda = m\omega^2$$

$$\left(-\lambda + \frac{11k}{8}\right)^2 - \frac{k^2}{64} = 0 \Rightarrow \lambda^2 - \frac{22k}{8}\lambda + \frac{120k^2}{64} = 0 \Rightarrow \lambda_{1,2} = \frac{\frac{22k}{8} \mp \sqrt{\frac{484k^2}{64} - \frac{480k^2}{64}}}{2}$$

$$\Rightarrow \lambda_{1,2} = \frac{\frac{22k}{8} \mp \frac{2k}{8}}{2} \Rightarrow \lambda_1 = \frac{3}{2}k \quad \lambda_2 = \frac{5}{4}k \Rightarrow \text{Normal frequencies } \omega = \sqrt{\frac{3k}{2m}} \ \& \ \sqrt{\frac{5k}{4m}}$$

Q.5) a) Distance = speed ⊗ time, but 120m is wrt lab frame and $\tau = 3\times10^{-7}$s is wrt $m_2$'s frame.

So, $v \neq \frac{120\,m}{3\times10^{-7}s}$ (that's greater than c anyway. I wonder if anyone will write $v_2 = 4\times10^{8}$ m/s!)

Choose the lab frame, $$\Delta t_{lab} = \frac{\Delta t_{m_2}}{\sqrt{1-\left(\frac{v_2}{c}\right)^2}} = \frac{d_{lab}}{v_2} \Rightarrow \frac{3\times10^{-7}}{\sqrt{1-\beta_2^2}} = \frac{120\,m}{\beta_2 \cdot c}$$

$$\Rightarrow \beta_2 = 0.8 \Rightarrow \underline{\underline{v_2 = 0.8c}} \qquad [\Delta t_{lab} = 5\times10^{-7}s]$$

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b) Momentum conservation $\Rightarrow p_3 = p_2 = \frac{m_{2,0} v_2}{\sqrt{1-\beta_2^2}} = \frac{90 \otimes 0.8}{0.6}$ MeV/c $= 120$ MeV/c

Energy conservation $\Rightarrow E_3 = M_0c^2 - E_2 = M_0c^2 - \frac{m_{2,0}c^2}{\sqrt{1-\beta_2^2}} = 350\,\text{MeV} - \frac{90}{0.6}\,\text{MeV} = 200\,\text{MeV}$

$$E_3^2 = p_3^2c^2 + m_{3,0}^2c^4 \Rightarrow \underline{\underline{m_{3,0} = 160\,\text{MeV}/c^2}}$$

$$E_3 = \frac{m_{3,0}c^2}{\sqrt{1-\beta_3^2}} \Rightarrow 200\,\text{MeV} = \frac{160\,\text{MeV}}{\sqrt{1-\beta_3^2}} \Rightarrow \beta_3 = 0.6 \Rightarrow \underline{\underline{v_3 = 0.6c}}$$

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c) Proper lifetime $$\Delta t = \Delta t_{lab} \otimes \sqrt{1-\beta_3^2} = \frac{120\,m}{0.6c}\sqrt{1-0.6^2} = \underline{\underline{5.33\times10^{-7}\,\text{sec}}}$$

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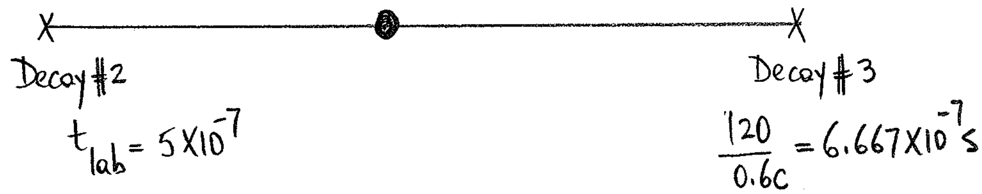
d) Since Decay #2 is casually related to Decay #1, there can be no reference in which decay #2 occurs before decay #1. Otherwise, casuality would be violated.

e)




#2 & #3 are not casually related $\Rightarrow$ possible to find a frame in which #2 & #3 occur simultaneously.

$\Delta t_{sym} = \dfrac{L\frac{v}{c^2}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$ Can be used. Here, $v$ is the relative velocity of the two frames. Events are simultaneous in one frame, in that frame the events are separated by $L \Rightarrow \Delta t$ gives the time difference between the events in the other frame.

Here, "the other frame" is lab $\Rightarrow \Delta t = 1.667\times10^{-7}$. $L$ = distance between #2 & #3 in the frame whose speed we want to know. Since $L_{lab} = 240$m $\Rightarrow L = 240\sqrt{1-\left(\frac{v}{c}\right)^2}$

$$\Rightarrow 1.667\times10^{-7} = \frac{240\sqrt{1-\left(\frac{v}{c}\right)^2}\,\frac{v}{c^2}}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \Rightarrow v = \frac{1.667\times10^{-7}}{240}3\times10^{8}\,c = \underline{\underline{\frac{5c}{24} \approx 0.208c}}$$

Since decay #3 occurs later in lab frame, this frame of relative speed 0.208c has to be moving <u>in the (+x) direction</u> to see #2 & #3 simultaneously.

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f) Easiest way to do this is to calculate velocity of $m_3$ wrt $m_2$.

$$v_{32} = \frac{0.8c + 0.6c}{1 + \frac{0.8\times0.6c^2}{c^2}} = 0.946c \,.\quad \Delta t_{3,0}\text{ (proper lifetime)} = 5.33\times10^{-7}\text{s}$$

$$\Rightarrow \Delta t_{3,2} \text{ is dilated by } \gamma \Rightarrow \Delta t_{3,2} = \frac{5.33\times10^{-7}\text{s}}{\sqrt{1-0.946^2}} = \underline{\underline{1.64\times10^{-6}\text{ s}}}$$