University of Illinois at Chicago
Department of Physics

Classical Mechanics
Qualifying Examination

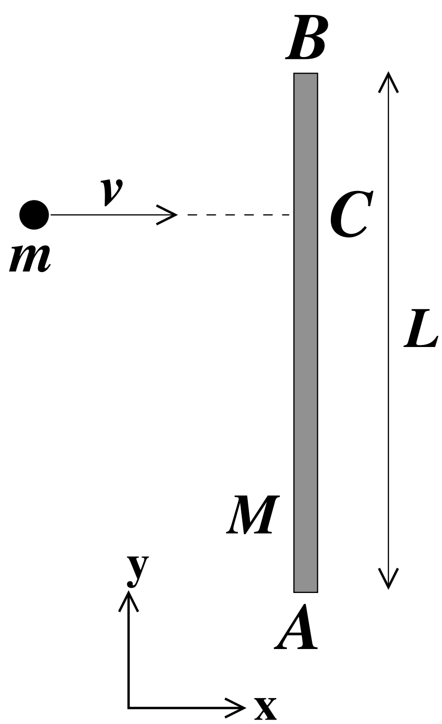
January 4, 2008
9.00 am – 12:00 pm

Full credit can be achieved from completely correct answers to <u>4 questions</u>. If the student attempts all 5 questions, all of the answers will be graded, and the <u>top 4 scores</u> will be counted toward the exam's total score.

1. A thin uniform rod of mass M and length $\overline{AB} = L$ lies on a horizontal surface, aligned along the y direction as shown below. An object of mass m moving along the x direction with a speed v collides with the rod at point C (The moment of inertia of a thin uniform rod with respect to its center of mass is $\frac{1}{12}ML^2$).

   (a) At what point should the object hit the rod so that immediately after the collision, the rod has an instantaneous axis of pure rotation around point A? Express your answer for $\overline{AC}$ in terms of L.

   (b) Now assume that the object m collides with the rod at a point C such that $\overline{AC} = 3L/4$ *and* the collision is elastic. After the collision, when the rod becomes aligned along the x direction for the first time, what is the distance between point B on the rod and the object with mass m? To make the calculations simpler, assume $m = M$ and express your answer in terms of L only.



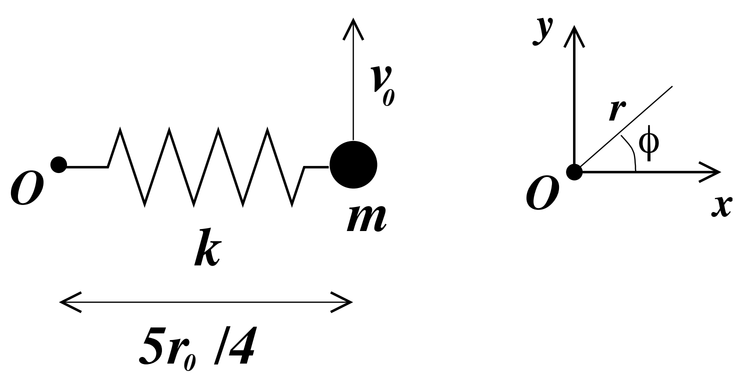
2. A massless spring of force constant k and natural length r_0 lies on a horizontal frictionless table. The spring is attached to the table at one end (the origin O), and can rotate freely around it. An object of mass m is attached to the other end of the spring.

(a) Assume that the spring is initially stretched along the x axis to a length of $5r_0/4$, and the object is given an initial speed of v_0 in the $+y$ direction. At what value of v_0 will the object rotate in a circle of fixed radius $5r_0/4$ about the origin?

(b) Using the polar coordinate system (r, ϕ), construct the Lagrangian of the system and

- Identify the cyclic coordinate and interpret the associated conserved quantity.
- Show that the total energy of the system can be expressed as $\frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$. Your expression for $V_{\text{eff}}(r)$ should contain only m, r, k, r_0 and the above conserved quantity.
- Sketch the potential $V_{\text{eff}}(r)$ and explain why there exists a stable circular orbit.
- If the radius of the stable circular orbit mentioned above is $r = 5r_0/4$, show that the magnitude of the conserved quantity becomes consistent with the result obtained in part (a).

(c) As the object moves in a fixed circle of radius $5r_0/4$, it is given a small additional push in the radial direction. Calculate the frequency of the resulting small radial oscillations in terms of k and m only.



3. The equation of motion for a particle of unit mass moving in a one-dimensional conservative force field is given by

$$\ddot{x} - \alpha x + \gamma x^3 = 0, \qquad \alpha, \gamma > 0.$$

The solution for the initial conditions $x(0) = \sqrt{\frac{2\alpha}{\gamma}}$ and $\dot{x}(0) = 0$ is given by (DO NOT PROVE THIS)

$$x(t) = \frac{\sqrt{\frac{2\alpha}{\gamma}}}{\cosh(t\sqrt{\alpha})}.$$

(a) Calculate the total energy of the object. Sketch the solution $x(t)$ and explain its asymptotic behavior in terms of the total energy.

(b) For a particle with the initial conditions $x(0) = \sqrt{\frac{\alpha}{\gamma}}\left(1 + \delta\right)$ and $\dot{x}(0) = 0$, find an approximate solution $x(t)$ valid for $\delta \ll 1$.

Now, consider the following equation of motion for a particle of unit mass:

$$\ddot{x} + \alpha x - \gamma x^3 = 0, \qquad \alpha, \gamma > 0.$$

(c) Find the solution $x(t)$ for the initial conditions $x(0) = \sqrt{\frac{2\alpha}{\gamma}}$ and $\dot{x}(0) = 0$.

(d) Starting with the above initial condition, calculate the time that it takes the particle to reach $x = +\infty$.

4. A particle of mass m is hung from a ceiling by two springs of negligible mass. The natural length of spring 1, with a force constant of k, is $L/2$. The natural length of spring 2, with an *unknown force constant*, is $L/3$. When the springs are connected to two points A and B on the ceiling which are separated by a distance $L\sqrt{2}$, the equilibrium lengths of both springs are observed to be L, as shown below. In this problem, choose the coordinate system, such that the particle in equilibrium is at the origin, and assume that the particle can only move in the (x, y) plane.

(a) Find an expression which relates the variables m, g, k and L.

The particle is now displaced to an arbitrary point P with position vector $\vec{r} = (x, y)$, where $r = |\vec{r}|$ is small. Let $\vec{R}_A$, $\vec{R}_B$ denote the position vectors of points A and B, respectively. By making a Taylor expansion to second order in $\vec{r}$, the new length $\overline{PA}$ of spring A can be shown to be (DO NOT PROVE THIS)

$$\overline{PA} = L\left(1 + \frac{r^2}{2L^2} - \frac{\vec{R}_A \cdot \vec{r}}{L^2} - \frac{(\vec{R}_A \cdot \vec{r})^2}{2L^4}\right)$$

with an analogous expression for $\overline{PB}$ with $\vec{R}_A$ replaced by $\vec{R}_B$.

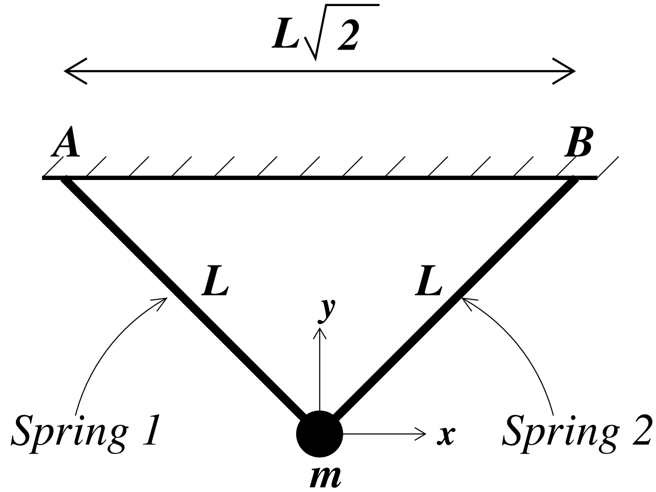
(b) Show that, for small r, the potential energies stored in the springs A and B are given by (keep only up to quadratic terms in $\vec{r}$)

$$U_A = \frac{k}{2}\left(\frac{L^2}{4} + \frac{r^2}{2} - \vec{R}_A \cdot \vec{r} + \frac{(\vec{R}_A \cdot \vec{r})^2}{2L^2}\right)$$

$$U_B = \frac{k}{2}\left(\frac{L^2}{3} + \frac{r^2}{2} - \vec{R}_B \cdot \vec{r} + \frac{(\vec{R}_B \cdot \vec{r})^2}{4L^2}\right)$$

(c) By first expressing the vectors $\vec{R}_A$ and $\vec{R}_B$ in the given coordinate system, construct the Lagrangian for this system. You should be able to express the Lagrangian in terms of x, y, $\dot{x}$, $\dot{y}$ and *only* the given parameters m, k, L, using your result in part (a).

(d) Set up Lagrange's equations of motion and calculate the normal frequencies for small oscillations.



5. An unstable subatomic particle has a rest mass $M_0 = 350$ MeV/c^2. With respect to the laboratory frame, the particle is at rest at the origin of coordinates. At $t = 0$ of the laboratory clocks, it decays (Decay #1) into two particles (particles 2 and 3) with rest masses of $m_{2,0}$ and $m_{3,0}$ moving away from the origin with speeds of v_2 and v_3 relative to the laboratory frame. Assume that particle 2 moves in the $-x$ direction as shown below.

Particles 2 and 3 are also unstable and decay (Decays #2 and #3) after they have moved equal distances of $d_2 = d_3 = 120$ meters, as measured by laboratory observers. It is known that the lifetime of particle 2 at rest is $\tau_2 = 3 \times 10^{-7}$ seconds, and its rest mass is $m_{2,0} = 90$ MeV/c^2.

(a) Calculate the speed v_2 (in units of c) of particle 2 relative to the lab frame after Decay #1.

(b) Calculate the rest mass $m_{3,0}$ (in MeV/c^2) and speed v_3 (in units of c) of particle 3 relative to the lab frame after Decay #1.

(c) Calculate the time it would take particle 3 at rest to decay.

(d) Is it possible to find an inertial frame of reference S' in which Decay #1 occurs *after* Decay #2? If so, calculate the minimum relative velocity (magnitude and direction) of S' with respect to the lab frame. If not, explain why.

(e) Is it possible to find an inertial frame of reference S'' in which Decay #2 and Decay #3 occur simultaneously? If so, calculate the relative velocity (magnitude and direction) of S'' with respect to the lab frame. If not, explain why.

(f) With respect to an observer in an inertial frame of reference attached to particle 2, how long does it take particle 3 to decay?